Physics 101 Homework 10 Solutions

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1 Ch. 11, §3, 5

\[
\frac{\Gamma(1/2)\Gamma(4)}{\Gamma(9/2)} = \frac{\Gamma(1/2)\Gamma(4)}{(7/2)(5/2)(3/2)(1/2)\Gamma(1/2)} = \frac{25\cdot 3!}{7(5)3} = \frac{32}{35}
\] (1.1)

2 Ch. 11, §3, 11

\[
\int_0^\infty x^5 e^{-x^2} \, dx = \frac{1}{2} \int_0^\infty u^2 e^{-u} \, du = \Gamma(3)/2 = 1
\] (2.1)

3 Ch. 11, §5, 5

\[
\Gamma(1/2 - n)\Gamma(1/2 + n) = \Gamma((1/2 + n))\Gamma(1 - (1/2 + n)) = \frac{\pi}{\sin(\pi(n + 1/2))} = (-1)^n \pi
\] (3.1)

\[
z!(−z)! = \Gamma(z + 1)\Gamma(1 - z) = z\Gamma(z)\Gamma(1 - z) = \frac{-\pi z}{\sin(\pi z)}
\] (3.2)

4 Ch. 11, §5, 6

\[
\frac{d^n}{dp^n} \Gamma(p) = \frac{d^n}{dp^n} \int_0^\infty x^{p-1}e^{-x} \, dx
\] (4.1)

\[
= \int_0^\infty \frac{d^n}{dp^n} x^{p-1-1}e^{-x} \, dx
\] (4.2)

\[
= \int_0^\infty (\log x)^n e^{(p-1)\log x}e^{-x} \, dx
\] (4.3)

\[
= \int_0^\infty x^{p-1}e^{-x}(\log x)^n \, dx
\] (4.4)

5 Ch. 11, §6, 1

\[
B(p, q) = \int_0^1 x^{p-1}(1 - x)^{q-1} \, dx = -\int_1^0 (1 - y)^{p-1}y^{q-1} \, dy = B(q, p)
\] (5.1)
\[ \Gamma(p, x) = \int_x^\infty t^{p-1} e^{-t} \, dt \]  
\[ = \left[ -t^{p-1} e^{-t} \right]_x^\infty + (p - 1) \int_x^\infty t^{p-2} e^{-t} \, dt \]  
\[ = x^{p-1} e^{-x} + (p - 1)x^{p-2} e^{-x} + (p - 1)(p - 2)x^{p-3} e^{-x} + \ldots \]  
\[ \]  
\[ \sqrt{n} \Gamma(n + 1) \sim \sqrt{\frac{2\pi n}{n}} n^{-n} = \sqrt{2\pi n} n^{n+1} e^{-n} \]  
\[ \Gamma(n + 3/2) \sim \sqrt{\frac{2\pi (n + 1/2)}{n}} (n + 1/2)^{n+1/2} e^{-n-1/2} \]  
\[ \Rightarrow \lim_{n \to \infty} \frac{\Gamma(n + 3/2)}{\sqrt{n} \Gamma(n + 1)} = \lim_{n \to \infty} \frac{\sqrt{2\pi (n + 1/2)n^{n+1} e^{-n} e^{-1/2}}}{\sqrt{2\pi n^{n+1} e^{-n}}} \]  
\[ = \lim_{n \to \infty} e^{-1/2} n^{n+1} \left( 1 + \frac{1}{n} \right)^{2n+2} \frac{1}{n^{n+1}} \]  
\[ = e^{-1/2} e^{1/2} = 1 \]  
\[ \lim_{n \to \infty} \frac{(n!)^{1/n}}{n} = \lim_{n \to \infty} \frac{\left( \sqrt{2\pi n} n^{-n} e^{-n} \right)^{1/n}}{n} \]  
\[ = \lim_{n \to \infty} \frac{\left( 2\pi n \right)^{1/2n} n}{n e^{\log(n)/2n + \log(e)/n}} = 1/e \]  
\[ \lim_{n \to \infty} n^x B(x, n) = \lim_{n \to \infty} \frac{n^x \Gamma(x) \Gamma(n)}{\Gamma(n + x)} \]  
\[ = \lim_{n \to \infty} \frac{\Gamma(x)n^x \sqrt{2\pi (n - 1)n^{-1} e^{-n+1}}}{\sqrt{2\pi (x + n - 1)x^{n-1} e^{-n-x+1}}} \]  
\[ = \lim_{n \to \infty} \frac{\Gamma(x)n^x \sqrt{n^{-1}(1 - 1/n)^{-n-1}}}{n^{n+x-1}(1 + (x - 1)/n)^{n+x-1}} \]  
\[ = \lim_{n \to \infty} \frac{\Gamma(x)n^x n^{-1} e^x e^{-1}}{n^{n+x-1} e^{-1}} = \Gamma(x) \]
\[\int_0^\infty \frac{e^{-t} dt}{1 + xt} = \left[ \frac{-e^{-t}}{1 + xt} \right]_0^\infty - \int_0^\infty \frac{xe^{-t} dt}{(1 + xt)^2} \] \hfill (10.1)

\[= 1 + \left[ \frac{xe^{-t}}{(1 + xt)^2} \right]_0^\infty + \int_0^\infty \frac{2x^2e^{-t} dt}{(1 + xt)^3} \] \hfill (10.2)

\[= 1 - x + \left[ \frac{2x^2e^{-t}}{(1 + xt)^3} \right]_0^\infty + \int_0^\infty \frac{6x^3e^{-t} dt}{(1 + xt)^4} \] \hfill (10.3)

\[= \sum (-1)^n n! x^n \] \hfill (10.4)