Problem 1 - Fraunhofer Pattern

To start, we note that we write $\gamma(x)$ as

$$\gamma(x) = \nabla \theta - \frac{2e}{\hbar} \int_1^2 \vec{A} \cdot d\vec{l}$$

(1)

In our case $\nabla \theta = \text{const}$ and $\vec{A} = -Bx \hat{z}$. Thus

$$\gamma(x) = \nabla \theta - \frac{2e}{\hbar} \int_0^d (-Bx)dz = \nabla \theta - \frac{2Bed}{\hbar}x$$

(2)

$$= \nabla \theta - \frac{2\pi Bd}{\varphi_0}x$$

(3)

Where $\varphi_0 = \frac{\hbar}{2\pi}$

The total flux through the josephson junction is

$$\Phi = \oint \vec{A} \cdot d\vec{l}$$

(4)

$$= \int_0^d A_z(x = L_x)dz + \int_d^0 A_z(x = 0)dz$$

(5)

$$= -BL_x(d) + 0$$

(6)

$$\Rightarrow \Phi = -BL_x d$$

(7)

Therefore

$$\gamma(x) = \Delta \theta + \frac{2\pi \Phi}{L_x \varphi_0}x$$

(8)

Now, we have that the total current is given by

$$I = \int_0^{L_x} j_c \sin(\gamma(x))$$

(9)

$$= \int_0^{L_x} j_c \sin \left( \Delta \theta + \frac{2\pi \Phi}{L_x \varphi_0}x \right)dx$$

(10)

$$= -j_c \frac{L_x \varphi_0}{2\pi \Phi} \left[ \cos \left( \Delta \theta + \frac{2\pi \Phi}{L_x \varphi_0}x \right) \right]_0^{L_x}$$

(11)

$$= j_c \frac{L_x \varphi_0}{2\pi \Phi} \left( \cos(\Delta \theta) - \cos \left( \Delta \theta + \frac{2\pi \Phi}{\varphi_0} \right) \right)$$

(12)
Now we write:

\[
\begin{align*}
\cos(\Delta \theta) - \cos(\Delta \theta + \alpha) &= \cos(\Delta \theta) - \cos(\Delta \theta) \cos(\alpha) + \sin(\Delta \theta) \sin(\alpha) \quad (13) \\
&= \cos(\Delta \theta)[1 - \cos(\alpha)] + \sin(\Delta \theta) \sin(\alpha) \quad (14) \\
&= \cos(\Delta \theta)[2 \sin^2(\alpha/2)] + 2 \sin(\Delta \theta) \sin(\alpha/2) \cos(\alpha/2) \quad (15) \\
&= 2 \sin(\alpha/2)[\cos(\Delta \theta) \sin(\alpha/2) + \sin(\Delta \theta) \cos(\alpha/2)] \quad (16) \\
&= 2 \sin(\alpha/2)[\sin(\Delta \theta + \alpha/2)] \quad (17)
\end{align*}
\]

Therefore, using this identity in our equation for current gives

\[
I = j_c \frac{L_x \varphi_0}{\Phi} \sin\left(\frac{\pi \Phi}{\varphi_0}\right) \sin\left(\Delta \theta + \frac{\pi \Phi}{\varphi_0}\right) \quad (18)
\]

Therefore, clearly the max current is:

\[
I = j_c \frac{L_x \varphi_0}{\Phi} \sin\left(\frac{\pi \Phi}{\varphi_0}\right) \quad (19)
\]

**Problem 2 - Hund’s Rule**

For an ion which has 4 electrons in its \textit{d} shell, Hund’s first rule tells us that we want to maximize the total spin. Since the \textit{d} shell has orbital quantum number \( \ell \in \{-2, 2\} \), there are five possible angular momentum states we can put the electron in. Then we can safely put all 4 electrons in the spin up \( s = +1/2 \) states so that total spin is \( S = 2 \).

Hund’s second rule states that we want to maximize total angular momentum \( L_z \). Each electron can have \( m_\ell \in \{-2, \ldots, 2\} \). Pauli’s exclusion principle states states that we can only have one spin up electron for each value of \( m_\ell \) so that the maximum total \( L_z \) that we can assign to the group of 4 electrons \( m_\ell = 2, 1, 0, -1 \). Then the overall angular momentum is \( L = 2 + 1 + 0 + (-1) = 2 \). Finally, the \textit{d} shell can hold 10 electrons, so it is less than half filled, so that Hund’s third rule states that \( J = |L - S| = |2 - 2| = 0 \).

Therefore \( S = 2, L = 2 \) and \( J = 0 \). Or in spectroscopic notation

\[
2S+1L_J = 5 P_0
\]

(Where \( P \) represents the \( L = 2 \) state in this notation).

Now, if we instead consider an ion with 2 \textit{f} electrons in its outer shell, in this case each angular quantum number can take on any value from \( \ell = \{-3, \ldots, 3\} \), so that the shell can hold 14 possible electrons. Then, Hund’s first rule implies that all electrons are spin up \( s = +1/2 \) so that \( S = 3/2 \).

Hund’s second rule says we want to assign the angular numbers \( m_\ell = 3, 2 \) and 1 to our three electrons so that the total angular momentum is \( L = 3 + 2 + 1 = 6 \).

Once again, our outer shell is less than half full so that Hund’s third rule states that \( J = |L - S| = |6 - 3/2| = 9/2 \).

Therefore our ground state would be the state with \( S = \frac{3}{2}, L = 6 \) and \( J = \frac{9}{2} \).

In spectroscopic notation (noting that \( L = 6 \) is represented by a capital \( I \)), this groundstate is denoted by

\[
2S+1L_J = 4 I_{9/2}
\]
Problem 3 - Brillouin Function

Part (a)

The energy of a spin $S$ moment in a magnetic field $H$ with g-factor $g$ is given by

$$E = -\vec{\mu} \cdot \vec{H} = m_s g \mu_B H$$

where $m_s \in \{S, S-1, \ldots, -S\}$

Then, the probability that the spin will be in a state with $m_s = m$ is given by

$$P(m) = \frac{e^{-E(m)/k_B T}}{\sum_{m=-S}^{S} e^{-E(m)/k_B T}}$$

Then, the average magnetization of the moment is then given by the average of the magnetic moment along the field directions $(m)$ time $g \mu_B$. That is $M = g \mu_B (m)$

Now, note that

$$\langle m \rangle = \sum_{m=-S}^{S} m P(m)$$

$$= \frac{\sum_{m=-S}^{S} m e^{-E(m)/k_B T}}{\sum_{m=-S}^{S} e^{-E(m)/k_B T}}$$

$$= \frac{\sum_{m=-S}^{S} m e^{-mg \mu H/k_B T}}{\sum_{m=-S}^{S} e^{-mg \mu H/k_B T}}$$

$$= \frac{\sum_{m=-S}^{S} m e^{mx}}{\sum_{m=-S}^{S} e^{mx}}$$

$$= \frac{d}{dx} \ln \left( \sum_{m=-S}^{S} e^{mx} \right)$$

where $x = -g \mu H/k_B T$.

Using mathematica we see that the sums in this expression are equal to

$$\sum_{m=-S}^{S} a^m = \frac{e^{-S(a^{2S+1} - 1)}}{a-1}$$

$$\Rightarrow \sum_{m=-S}^{S} e^{mx} = \frac{e^{-Sx(e^{2S+1}x - 1)}}{e^x - 1} = \frac{e^{(S+1)x} - e^{-Sx}}{e^x - 1}$$

$$= \frac{e^{(S+\frac{1}{2})x} - e^{-(S+\frac{1}{2})x}}{e^{x/2} - e^{-x/2}}$$

$$= \frac{\sinh((S + \frac{1}{2})x)}{\sinh(x/2)}$$

Therefore

$$\langle m \rangle = \frac{d}{dx} \ln \left( \frac{\sinh((S + \frac{1}{2})x)}{\sinh(x/2)} \right)$$

$$= \frac{\sinh(x/2) \left[ (S + \frac{1}{2}) \cosh((S + \frac{1}{2})x) \right]}{\sinh((S + \frac{1}{2})x) \left[ \sinh(x/2) \right]} - \frac{\sinh((S + \frac{1}{2})x)}{2 \sinh^2(x/2) \cosh(x/2)}$$

$$= (S + \frac{1}{2}) \coth((S + \frac{1}{2})x) - \frac{1}{2} \coth(x/2)$$
Now, write our answer in terms of \( x' = g\mu_B SH/k_B T = -Sx \), then

\[
M = -g\mu_B S\langle m \rangle = g\mu_B S \left[ \frac{2S + 1}{2S} \coth \left( \frac{2S + 1}{2S} x' \right) - \frac{1}{2S} \coth \left( \frac{x'}{2S} \right) \right]
\]

Therefore

\[
M = g\mu_B S \left( g\mu_B SH/k_B T \right)
\]

Part (b)

Let \( a = (2S + 1) \) and \( b = 1/(2S) \). Then

\[
B_S(x) = ab \coth(abx) - b \coth(bx) \approx 1 - 1 + \frac{(ab)^2x}{3} - \frac{b^2x}{3} \quad (35)
\]

\[
= \frac{1}{3} b^2 x (a^2 - 1) \quad (36)
\]

\[
\Rightarrow \quad \frac{d B_S(x)}{dx} \bigg|_{x \to 0} = \frac{1}{3} (a^2 - 1) b^2 = \frac{1}{3} \left( \frac{(2S + 1)^2 - 1}{4S^2} \right) = \frac{1}{3} \frac{S + 1}{S} \quad (37)
\]

Then,

\[
\frac{\partial M}{\partial H} \bigg|_{H \to 0} = g\mu_B S \frac{\partial B_S(x)}{\partial x} \bigg|_{x \to 0} \frac{\partial x}{\partial H} \quad (38)
\]

\[
= g\mu_B S \left( \frac{S + 1}{3S} \right) \frac{g\mu_B S}{k_B T} \quad \text{using } x = g\mu_B SH/k_B T \quad (39)
\]

Therefore

\[
\chi = \frac{\partial M}{\partial H} \bigg|_{H = 0} = \frac{(g\mu_B)^2 S(S + 1)}{3k_B T} \quad (40)
\]