Problem 1 - Non-Linear $M(H)$

The Hamiltonian of a Heisenberg ferrromagnet in a magnetic field $H$ is given by

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - g\mu_B \sum_i \mathbf{H} \cdot \mathbf{S}_i , \quad J > 0$$ (1)

It was seen in class that in mean field theory we can study the system

$$H = -J \sum_{\langle ij \rangle} \left( \mathbf{S}_i \cdot \mathbf{S}_j + \mathbf{S}_i \cdot \langle \mathbf{S}_j \rangle - \langle \mathbf{S}_i \rangle \cdot \langle \mathbf{S}_j \rangle \right) - g\mu_B \sum_i \mathbf{H} \cdot \mathbf{S}_i$$ (2)

$$= -zJ \sum_i \left( \mathbf{m} \cdot \mathbf{S}_i + \text{const.} \right) - g\mu_B \sum_i \mathbf{H} \cdot \mathbf{S}_i$$ (3)

$$= - \sum_i (zJm + g\mu_B H) \cdot \mathbf{S}_i$$ (4)

where $\mathbf{m} = \langle \mathbf{S}_i \rangle$ and $z$ equals the number of nearest neighbour lattice sites.

We saw in class that the magnetization for a spin-1/2 particle interacting with an effective field $h$ is given by the solution to the equation

$$m = \frac{1}{2} \tanh \left[ \frac{h}{2kT} \right]$$ (5)

We know that the spins will align in the same direction as the magnetic field so that $\mathbf{m}$ and $\mathbf{H}$ point in the same direction.

Then, in our system, the effective field is given by $h = zJm + \zeta g\mu_B |H|$, where $\zeta = \text{sign}(H)$. Therefore, the magnetization is given by the solution to the equation

$$m = \frac{1}{2} \tanh \left[ \frac{zJm + \zeta g\mu_B |H|}{2kT} \right]$$ (6)

We also saw in class that the critical point for the Heisenberg ferromagnet in mean field theory is at $kT_c = zJ/4$. Subbing this in for our temperature gives

$$m = \frac{1}{2} \tanh \left[ \frac{2(zJm + \zeta g\mu_B |H|)}{zJ} \right] = \frac{1}{2} \tanh \left[ \frac{2m + 2\zeta g\mu_B |H|}{zJ} \right]$$ (7)

When $H$ is small, we know that the magnetization $m$ is also small near the critical point $T_c$, since the magnetization is zero above the critical temperature and the magnetization is a continuous function of temperature for this model. Therefore we can Taylor expand our $\tanh(x) \approx x - x^3/3$ function
\[ m = \frac{1}{2} \left[ 2m + \frac{2 \zeta \mu_B |H|}{zJ} + \frac{1}{3} \left( 2m + \frac{2 \zeta \mu_B |H|}{zJ} \right)^3 \right] \]

\[ 0 = \frac{\zeta \mu_B |H|}{zJ} + \frac{1}{6} \left( 2m + \frac{2 \zeta \mu_B |H|}{zJ} \right)^3 \]

\[ \Rightarrow 2m \approx \left( \frac{6 \zeta \mu_B |H|}{zJ} \right)^{1/3} + \mathcal{O}(H) \]

\[ m = \text{sign}(H) \frac{1}{2} \left( \frac{6 \zeta \mu_B}{zJ} \right)^{1/3} |H|^{1/3} \]

Where above I expanded the equation above to lowest nonvanishing order in both \( m \) and \( H \).

Therefore

\[ M = g \mu_B m = \text{sign}(H) \frac{g \mu_B}{2} \left( \frac{6 \zeta \mu_B}{zJ} \right)^{1/3} |H|^{1/3} \]
Problem 2 - Quantum transverse-field Ising model

Part (a)
We start with the spin 1/2 Hamiltonian, and apply the mean field approximation:

\[ H = -J \sum_{\langle ij \rangle} S_i^z S_j^z - h_\perp \sum_i S_i^x \]

\[ \approx -J \sum_{\langle ij \rangle} \langle S_i^z \rangle S_j^z - \langle S_i^z \rangle \langle S_j^z \rangle - h_\perp \sum_i S_i^x \]

\[ = \sum_i (-z J m) S_i^z - h_\perp S_i^x + \text{const} \]

\[ = -\sum_i h \cdot S \quad \text{with} \quad h = (h_\perp, 0, z J m) \]

where \( m = \langle S_i^z \rangle \). Therefore, our mean field Hamiltonian can be written as a set of spins in an effect field \( h_{\text{eff}} = (h_\perp, 0, z J m) \)

Part (b)
Now, we assume that we are at zero temperature and that each spin is aligned fully with the effect field \( h_{\text{eff}} \).

Therefore, we now that \( S \) points in the direction \((h_\perp, 0, J z m)\). We also know that \(|S| = S = 1/2\). Then, we can write

\[ S_i = \frac{1}{2} \frac{J z m \hat{z} + h_\perp \hat{x}}{\sqrt{J^2 z^2 m^2 + h_\perp^2}} \]

Since this is the proper normalized spin-1/2 vector which points in the \( h_{\text{eff}} \) direction.

Since all spins point in the same direction, the magnetization is just the \( z \) component of this spin vector

\[ m = \langle S_i^z \rangle = \frac{1}{2} \frac{J z m}{\sqrt{J^2 z^2 m^2 + h_\perp^2}} \]

\[ \Rightarrow m^2 = \frac{J^2 z^2 m^2}{4(J^2 z^2 m^2 + h_\perp^2)} \]

\[ 0 = 4J^2 z^2 m^4 + (4h_\perp^2 - J^2 z^2) m^2 \]

Part (c)
Therefore, for a given transverse field \( h_\perp \), the magnetization is given by the solution to the equation

\[ 4J^2 z^2 m^4 + 42h_\perp^2 - J^2 z^2) m^2 = 0 \]

Solving this gives

\[ m^2(4J^2 z^2 m^2 + (4h_\perp^2 - J^2 z^2)) = 0 \]

\[ \Rightarrow m = 0 \quad \text{or} \quad m = \pm \sqrt{\frac{1}{4} - \frac{h_\perp^2}{J^2 z^2}} \]

We see that for \( \frac{h_\perp^2}{J^2 z^2} > \frac{1}{4} \), the only solution to this equation is at \( m = 0 \). Therefore the critical field \( h_c \) occurs when \( \frac{h_\perp^2}{J^2 z^2} = \frac{1}{4} \) (i.e when \( h_\perp = J z / 2 \)). When \( h_\perp > h_c \), we have that \( m(h_\perp) = 0 \)

Therefore

\[ m(h_\perp) = \pm \sqrt{\frac{1}{4} - \frac{h_\perp^2}{J^2 z^2}} \quad \text{for} \quad h_\perp < h_c \quad \text{where} \quad h_c = \frac{J z}{2} \]
\[
\langle S^z_{\text{TOT}} \rangle / N = +S - \int_{\varepsilon=0}^{\infty} n_B(\varepsilon) d\varepsilon
\]  
(25)

The number of magnons at a given temperature \( T \) is given by

\[
\sum_k n_k = \int d\varepsilon D(\varepsilon) \langle n(\varepsilon) \rangle
\]  
(27)

The factor of \( D(\varepsilon) \) is the number of frequency modes \( k \) which exist at a given energy \( \omega \). For a given wavevector \( k \), we have the number of states within a sphere of radius \( k \) is \( N(k) = (2\pi)^{-3}(\frac{4}{3}\pi k^3) \). Then the number of states within a shell of thickness \( d\varepsilon \) is:

\[
D(\varepsilon)d\varepsilon = \frac{dN(k)}{dk} \frac{dk}{d\varepsilon} = \frac{1}{(2\pi)^3} 4\pi k^2 \frac{dk}{d\varepsilon} d\varepsilon
\]  
(28)

Therefore

\[
\langle S^z_{\text{TOT}} \rangle / N = +S - 4\pi \int_{k=0}^{k_{\varepsilon}} \frac{k^2}{e^{\beta c k^2} - 1} dk
\]  
(30)

\[
\approx +S - \frac{1}{2\pi^2} \int_{0}^{\infty} \frac{k^2}{e^{\beta c k^2} - 1} dk
\]  
(31)

where we make the approximation that the upper bound on the integral goes to infinity since \( n_B(k) \to 0 \) exponentially fast as \( k \) becomes large.

Now, let \( x = \beta c k^2 \) So that \( dx = 2\beta c k dk \) and we can write the integral above as

\[
\langle S^z_{\text{TOT}} \rangle / N = +S - \frac{1}{2\pi^2} \int_{0}^{\infty} \frac{dx}{2\beta c \sqrt{3} c e^{\frac{x}{2\beta c}} - 1}
\]  
(33)

\[
= S - \frac{1}{4\pi^2 c} (kT)^{3/2} \int_{0}^{\infty} \frac{x}{e^{x} - 1} dx
\]  
(34)

\[
= S - \frac{1}{12} \frac{(kT)^{3/2}}{c}
\]  
(35)

where in the last line I used the fact that the integral over \( x \) is equal to \( \pi^2 / 6 \).