Problem 1

To solve this problem we convert the energies into velocities using the relativistic dispersion relation (see equation (5.47) of Hartle)

\[ E^2 = m^2 + \left| \vec{p} \right|^2 = m^2 + \frac{m^2 V^2}{1 - V^2} \Rightarrow V = \sqrt{\frac{1}{1 + \left( \frac{m}{E} \right)^2}}, \]  

(1)

where \( V = \| \vec{V} \| \). Plugging in \( m \approx 938 \text{MeV} \) for the proton we get \( V_{\text{Tevatron}} \approx 0.9999998900 \) and \( V_{\text{LHC}} \approx 0.9999999910 \). Converting the difference to mph we get \( V_{\text{LHC}} - V_{\text{Tevatron}} \approx 68 \text{mph} \).
Solution:

a) \((gt)^2 < 1 + (gt)^2\) so \(dx/dt < 1\).

b)

\[
\begin{align*}
\mathbf{u}^t &= \frac{1}{\sqrt{1 - V^2}} = \sqrt{1 + (gt)^2} \\
\mathbf{u}^x &= \frac{V}{\sqrt{1 - V^2}} = g t \\
\mathbf{u}^y &= \frac{u^z}{\sqrt{1 - V^2}} = 0
\end{align*}
\]

c) The clock of an observer riding on the particle reads proper time. The proper time elapsed from \(t = 0\) to \(t\) is

\[
\tau = \int_0^t \frac{dt}{\sqrt{1 - V^2}} = \int_0^t \frac{dt}{\sqrt{1 + (gt)^2}} = \frac{1}{g} \sinh^{-1}(gt). \tag{1}
\]

The particle trajectory is

\[
x(t) - x_0 = \int_0^t \frac{dt}{\sqrt{1 + (gt)^2}} = \frac{1}{g} \sqrt{1 + (gt)^2}.
\]

Thus the relation between \(\tau\) — the time on the observer’s clock — and the location \(x\) is

\[
(x - x_0) = \frac{1}{g} \sqrt{1 + \sinh^2(g\tau)} = \frac{1}{g} \cosh(g\tau). \tag{2}
\]

d) The four force is \(f^\alpha = m d^2 x^\alpha / d\tau^2\) or

\[
f^\alpha = (mg \sinh(g\tau), mg \cosh(g\tau), 0, 0).
\]

The three force is given by \(\mathbf{F} = m d\mathbf{u}/dt\)

\[
F^t = (mg, 0, 0).
\]
12. The 2 mile long Stanford linear accelerator accelerates electrons to an energy of 40 GeV as measured in the frame of the accelerator. Idealize the acceleration mechanism as a constant electric field \( E \) along the accelerator and assume that the equation of motion is

\[
\frac{d\vec{p}}{dt} = eE.
\]

Here \( \vec{p} \) is the spatial part of the relativistic momentum \( p \).

Assuming that the electron starts from rest, find its position along the accelerator as a function of time in terms of its rest mass \( m \) and \( F \equiv e|\vec{E}| \).

What value of \( |\vec{E}| \) would be necessary to accelerate it to its final energy.

\[\text{Solution:}\]

The momentum \( p \) increases linearly with time. Denoting \( e|\vec{E}| \) by \( F \), we have

\[ p(t) = Ft \]

if \( t = 0 \) is the time the particle is at rest. From the momentum we can find the three-velocity \( V \). From (5.45)

\[
\frac{dx}{dt} = V = \frac{p}{E} = \frac{p}{\sqrt{m^2 + p^2}} = \frac{Ft}{\sqrt{m^2 + (Ft)^2}}.
\]

Integrating this relation gives

\[
x(t) = F^{-1}\sqrt{m^2 + (Ft)^2} - \frac{r(0)}{V}.
\]

for the distance from the starting point as a function of time.
b) The energy reached at time $t$ is

$$E = \sqrt{m^2 + p^2(t)} = \sqrt{m^2 + (Ft)^2}. \quad (3)$$

So, from (2), the energy reached at a distance $x$ from the start is

$$E(t) = Fx(t). \quad (4)$$

Evaluating this at the final time when $E = 40 \text{ GeV}$ and $x = 2 \text{ mi}$ gives

$$F = \frac{40 \text{ GeV} \times (1.6 \times 10^{-10} \text{ J/GeV})}{2 \text{ mi} \times (1609 \text{ ft/ mi})} = 1.99 \times 10^{-12} \text{ J/m}. \quad (5)$$

The charge on the electron is $1.6 \times 10^{-19} \text{ Coul}$. The required field is therefore

$$|E| \approx 12 \text{ million Volts/m}. \quad (6)$$

5-13. [B, S] One reaction for photoproducing pions is

$$\gamma + p \rightarrow n + \pi^+.$$

Find the minimum energy (the threshold energy) a photon would have to have to produce a pion in this way in the frame in which the proton is at rest. Is this energy within reach of contemporary accelerators?

Solution: The threshold condition, Ref. (b) in Box 5.1 just needs to be evaluated in the frame in which the proton is at rest. In that frame

$$p_p^\alpha = (m_p, 0, 0, 0)$$

and the condition gives

$$-2E_\gamma m_p - m_p^2 = - (m_n + m_\pi)^2$$

for the threshold energy $E_\gamma$ of the photon. Solving for $E_\gamma$ gives the approximation for $m_n \approx m_p$

$$E_\gamma = m_\pi \left(1 + \frac{m_\pi}{2m_p}\right) \approx 150 \text{ MeV}.$$
**Solution:** In any inertial frame we have from (5.43) and (5.17c):

\[ p \cdot p = -E^2 + \vec{p} \cdot \vec{p} . \]  \hspace{1cm} (1)

In the inertial frame in which the observer is at rest at the time of the measurement, \( E \) is the measured energy, and \( |\vec{p}| \) is the measured magnitude of the momentum. Then

\[ |\vec{p}| = (E^2 + p \cdot p)^{1/2} . \]  \hspace{1cm} (2)

But also from (5.83) the measured energy \( E \) is

\[ E = -p \cdot u_{\text{obs}} . \]  \hspace{1cm} (3)

Substituting (3) into (2) gives the required result.