a) If \( a(t) = \left( \frac{t}{t_*} \right)^{\frac{1}{2}} \) then the Hubble "constant" at any time \( t \) is

\[
H = \frac{a}{a} = \frac{1}{2t}.
\]

At the present age, \( t_0 = 14 \times 10^9 \) years, we have

\[
H_0 = \left( 28 \times 10^9 \text{ yr} \right)^{-1} = 3.6 \times 10^{-11} \text{ yr}^{-1}.
\]

b) The temperature of radiation varies with \( a \) as

\[
T = T_0 \left( \frac{a_0}{a} \right) = T_0 \left( \frac{t_0}{t} \right)^{\frac{1}{2}}.
\]

Let \( t_{3k} \) be the time of decoupling when \( T = 3000^\circ K \). The present temperature is \( 3^\circ K \), so we have

\[
3000^\circ K = 3^\circ K \left( \frac{14 \times 10^9 \text{ yr}}{t_{3k}} \right)^{\frac{1}{2}}
\]

so

\[
t_{3k} = 14 \times 10^3 \text{ yr}.
\]

18-3. Consider a flat FRW model whose metric is given by (18.1). Show that, if a particle is shot from the origin at time \( t_* \) with a speed \( V_* \) as measured by a co-moving observer (constant \( x, y, z \)), then asymptotically it comes to rest with respect to a co-moving frame. Express the co-moving coordinate radius at which it comes to rest as an integral over \( a(t) \).

Solution: Orient coordinates so that the particle is moving along the \( x \)-axis and restrict attention to the two relevant dimensions \( (t, x) \). The metric is [cf. (18.1)]

\[
ds^2 = -dt^2 + a^2(t) \, dx^2.
\]

This is unchanged under displacements in \( x \). There is thus a Killing vector \( \xi^a = (0, 1) \) and a conserved quantity \( \xi \cdot u \equiv q \) which is

\[
\xi \cdot u = a^2(t) \frac{dx}{d\tau} = q
\]
where \( \mathbf{u} \) is the particle’s four-velocity. The value of \( q \) is determined by the initial velocity \( V_* \) (see below). Another integral is supplied by the normalization condition

\[
\mathbf{u} \cdot \mathbf{u} = -\left( \frac{dt}{d\tau} \right)^2 + a^2(t) \left( \frac{dx}{d\tau} \right)^2 = -1. 
\]  

(3)

Eqs. (2) and (3) can be solved for the components of the four-velocity

\[
u^x = \frac{dx}{d\tau} = \frac{q}{a^2(t)}, \quad u^t = \frac{dt}{d\tau} = \left[ 1 + \frac{q^2}{a^2(t)} \right]^{-\frac{1}{2}}.
\]  

(4)

The three-velocity \( dx/dt \) is then

\[
\frac{dx}{dt} = \frac{u^x}{u^t} = \frac{q}{a^2(t)} \left[ 1 + \frac{q^2}{a^2(t)} \right]^{-\frac{1}{2}}.
\]  

(5)

As the universe expands, \( a(t) \) grows and \( dx/dt \) tends to zero. The particle therefore comes to rest at a coordinate \( x_f \) which can be found by integrating (5):

\[
x_f = \int_{t_i}^{t_f} dt \frac{q}{a^2(t)} \left[ 1 + \frac{q^2}{a^2(t)} \right]^{-\frac{1}{2}}.
\]  

(6)

It remains to express \( q \) in terms of the initial velocity \( V_* \) measured by a co-moving observer. Orthonormal basis vectors for such an observer are \( \mathbf{e}_1 = (1, 0) \) and \( \mathbf{e}_2 = [0, 1/a(t_*)] \). We have, for instance [cf. (5.82)],

\[
u^x = \mathbf{e}_2 \cdot \mathbf{u} = a(t_*) \frac{dx}{d\tau} = \frac{q}{a(t_*)},
\]  

(7)

where the last equality follows from (2). Thus,

\[ q = a(t_*) v^x = a(t_*) \frac{V_*}{\sqrt{1 - V_*^2}}. \]  

(8)

18-4. [S] Suppose the present value of the Hubble constant is 72 (km/s)/Mpc and that the universe is at critical density. A photon is emitted from our galaxy
18-19. (de Sitter Space) Solve the Friedman equation (18.63) for the scale factor as a function of time for closed FRW models that have only vacuum energy $\rho_v$. Do these models have an initial big-bang singularity?

**Solution:** Defining $H$ in terms of the vacuum energy of (18.40), the Friedman equation (18.63) for closed FRW models ($k = 1$) is

$$\dot{a}^2 - H^2 a^2 = -1.$$  

Reorganizing, this is

$$\frac{da}{(H^2 a^2 - 1)^{1/2}} = dt.$$  

Integrating both sides gives

$$a(t) = \frac{1}{H} \cosh(Ht)$$

with suitable choice for the origin of $t$.

Evidently there is no big bang singularity since $a(t)$ never vanishes over the whole range of $t$. Rather the universe starts from large values of $a$ at large negative times, collapses, reaches a minimum value of $a$ at $t = 0$, and re-expands. This solution is called De Sitter space.
18-23. (a) Show that for FRW models with any combination of matter and radiation but no vacuum energy, the curve of \( a(t) \) curves downward, i.e. has negative second derivative. Show that this means that \( 1/H_0 \) is always larger than the age \( t_0 \).

(b) Show that this is not always the case if there is a non-zero vacuum energy.

Solution: (a) Solving for \( d\ddot{a}/d\ddot{a} \) in (18.77) and differentiating with respect to
\( \dot{t} \) gives
\[
\frac{d^2 \dot{a}}{dt^2} = -\frac{\Omega_r}{\dot{a}^2} - \frac{\Omega_m}{2\dot{a}} + \Omega_v \dot{a} . 
\] (1)

The \( \Omega \)'s are positive because the densities of matter and radiation are positive. If \( \Omega_v = 0 \), the right hand side of the above relation is negative, whence \( \ddot{a} < 0 \). The universe decelerates. As Figure 18.2 makes clear, when the curve of \( a(t) \) curves downward, \( 1/H_0 \) is an upper bound on the age of the universe.

(b) If \( \Omega_m = \Omega_r = 0 \), and \( \Lambda > 0 \), then \( \Omega_v \) is positive, \( \ddot{a} \) is positive and the universe accelerates.

18-24. (The Einstein Static Universe) Consider a closed \( (k = +1) \) FRW model containing a matter density \( \rho_m \), a vacuum energy density corresponding to a positive cosmological constant \( \Lambda \), and no radiation.

(a) Show that for a given value of \( \Lambda \) there is a critical value of \( \rho_m \) for which the scale factor does not change with time. Find this value.

(b) What is the spatial volume of this universe in terms of \( \Lambda \)?

(c) If \( \rho_m \) differs slightly from this value the scale factor will vary in time. Does the evolution remain close to the static universe or diverge from it.

Comment: This is the Einstein static universe for which Einstein originally introduced the cosmological constant.

Solution:

a) There is a stationary solution for \( a(t) \) at the maximum of the effective potential \( U_{\text{eff}}(\dot{a}) \) defined in (18.78). This occurs when

\[ -\frac{\Omega_m}{\dot{a}^2} + 2\Omega_v \dot{a} = 0 \]

or

\[ \rho_m = 2\rho_v = \frac{\Lambda}{4\pi} . \]

b) The Friedman equation (18.77) implies

\[ U_{\text{eff}}(\dot{a}) = \frac{\Omega_c}{2} \]
or
\[-\frac{8\pi}{3} (\rho_m + \rho_v) a^2 = -1\]
or
\[a = \frac{1}{\sqrt{\Lambda}}.\]

So the volume is [cf. (18.55)]
\[V = 2\pi^2 a^3 = 2\pi^2 \Lambda^{-\frac{3}{2}}.\]

c) As the plot of \(U_{\text{eff}}(a)\) in Figure 18.9 shows, the static universe is unstable. A small change in \(\rho_m\) will either cause it to expand to infinite volume or collapse to a singularity.