1. This material where OsB takes place is Er₂Ti₂O₇.

2. Er is a rare-earth (RE).

3. Er³⁺ is magnetic 4f³⁻ will return to this below.

4. Ti is unimportant.

5. "Mediates" the interactions between the Er moments.

6. Er sits on a pyrochlore lattice.

7. To recall spin ice: same chemical stoichiometry.

8. This means there are unpaired electrons (unfilled shell).

9. \( L = 6 \)

10. \( S = \frac{3}{2} \)

11. Er is heavy (bottom of the periodic table).

12. Large spin-orbit coupling = \( L \cdot S \) (relativistic effect).

13. Good QN is \( J = L + S \).

14. Florio: manifold selected by Hund's rules: \( J = \frac{5}{2} \) or \( \frac{3}{2} \).

15. Crystal fields or electric fields of the solvent atoms (protons mostly) split (couple) levels.

16. The degeneracy of the manifold acts on the orbital part.

17. S capture Kramers degeneracy (half-integer J).

18. Result: ugly electron paramagnetic resonance.

19. \( J = \frac{7}{2} \)K !

20. Effective spin \( \frac{1}{2} \).

21. Call that spin \( \frac{1}{2} \).

22. Now we want to ask: What's the Hamiltonian? We need it to make progress (specific to ETO at least).
* Spin interactions: \[ \frac{1}{2} \sum_{j} \frac{1}{J} \left( \Sigma S_j^x \Sigma S_j^y \right) = H \]

* Consider informed choice for translational invariance:

\[ S = \cos \alpha \cos \phi \left( \Phi^+ \Phi - \frac{1}{2}I \right) \]  
\[ \Phi = \rho e^{i \alpha} \]

* Write \[ F = \Phi \Phi^* = \Phi^2 \]

\[ \Phi \] will be of the form:

\[ \Phi = \rho e^{i \alpha} \]

* There is not just TR, but also geometric symmetries:

\[ \Phi \rightarrow \Phi \rightarrow \Phi e^{i \frac{2\pi}{3}} \]

\[ \alpha \rightarrow \alpha + \frac{2\pi}{3} \]

* A true (not easy to see!) for the others:

\[ \Phi = \rho e^{i \alpha} \rightarrow \Phi e^{i \frac{2\pi}{3}} \]

\[ \alpha = \alpha^* = 0 \]

* \( F \) is independent of \( \alpha \):

\[ U(1) \] symmetry

\[ \text{(circle)} \]

Note: * Used only discrete symmetries to prove \( C^0 \) degeneracy

= "accidental" degeneracy

* Symmetry-protected degeneracy, did not use any specific coefficients.

Consequences: "Goldstone mode" (co-syn)

\[ \psi \] scalar

\[ \bar{\psi} \psi \] gapless

\[ \rho^2 \text{ can write (and we free energy) is zero for good}

\[ S = \int d^3 x \left( \partial_x \Phi \right)^2 + \frac{\kappa}{2} \left( \partial_x \Phi \right)^2 \]

\[ \partial^6 \text{ order terms will come in }

\[ \frac{1}{2} \kappa \frac{\partial^6 \Phi}{\partial \phi^6} \text{ symmetries of}

\[ \phi \text{ (if } \Phi \text{ origin)
Indeed, ground (stable) state.

For this need the specific Hamiltonian

- To said 2-spin unit $J_{ij} S_i^a S_j^a$

- constrained by symmetry is not $3 \times 3 \times 4 \times 4 = 144$ by just 4$

symmetries turn the bonds into one another

so knows one term already if studied $\Delta \alpha$

spin ice: $H = J_{\alpha} \sum_{i>j} S_i^a S_j^a$

local basis

To get the coefficients by fits to neutron scattering, will return to this later

To get the $\omega$'s using spin wave theory -> HP other sheet.

the $\omega(k)$'s

- depend on $\alpha$ (just like in the previous example where $\omega_k(\alpha)$

- the zero point energy $= \sum_k \frac{e^{i \omega_k}}{2}$

- $E \sim \frac{1}{2} \cos 6\alpha$

- $\Delta > 0$
can write a "low energy (effective) action for the system at $T = 0$".

Assume slow time & space variations of $\omega$.

$$S = \int \frac{d^3x}{(2\pi)^3} \left[ \sum_{\nu} \left( \frac{1}{2} \left( \partial_\mu \omega \right)^2 + \frac{1}{2} \left( 2 \omega \right)^2 - \frac{1}{2} \cos 6\omega \right) \right]$$

Note one can build characteristic length & velocity

$$\gamma \equiv \sqrt{\frac{K_1}{18\lambda}} \quad \omega_0 = \sqrt{\frac{K_1}{2\lambda}}$$

What's the gap?

$$\alpha(x, y) \rightarrow$$

Go to Fourier space.

$$S = \frac{1}{2\beta} \sum_{\nu} \left[ \sum_{k, \omega} \left( \omega - \omega_0 \right) \chi_k \chi_{-k} \omega \left( \omega^2 + \omega_0^2 \right) \right]$$

The "gap" is the lowest from $\omega_0$.

Energy $\rightarrow \sqrt{\frac{18\lambda}{\pi}}$.

$\alpha(x, y) \rightarrow$$

Go for theorists: it's the "mass" i.e. coefficient of the $\alpha^2$ term.

Go note: periodicity of some unimportant have some look at low energies.
Need spin polarization for this to be reasonable. Assume "transformed invariance". Write fluctuations as follows:

\[ \tilde{\mathbf{a}} \text{ is along the classical (or TF) direction (select by the energy or the free energy)} \]

\[
\begin{align*}
\tilde{S} \cdot \tilde{\mathbf{a}} &= S - n \tilde{a} \\
\tilde{\mathbf{S}} \cdot \tilde{\mathbf{a}} &= \sqrt{s} x \tilde{a} \\
\tilde{\mathbf{S}} \cdot \tilde{\mathbf{a}} &= \sqrt{s} y \tilde{a}
\end{align*}
\]

\[ (\tilde{\mathbf{a}}, \tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \text{ is an orthonormal basis} \]

\[ (\tilde{x}, \tilde{y}) = i \lambda \]

Note: 8x8 matrices will be involved.

Expand in large \( s \), \( s = \frac{1}{2} \) but it usually works really well!

**Classical Energy**

\[ E = \tilde{S}_i \cdot \tilde{J}_i \cdot \tilde{S}_j \]

\( \tilde{J} \) involves the \( \tilde{J}_{22}, \tilde{J}_3, \tilde{J}_4 \), etc.

\[ \text{e.g. } J_{01} = \begin{pmatrix} J_2 & J_4 & J_6 \\ J_4 & J_2 & J_6 \\ J_6 & J_4 & J_2 \end{pmatrix} \]

\[ J_1 = J_6 \\
J_2 = J_4 \\
J_3 = J_4 \\
J_4 = J_6 \]

Go to Fourier space:

\[ \text{get } H = \frac{1}{\hbar} \begin{pmatrix} X^T \text{ Y}^T \end{pmatrix} \begin{pmatrix} A_2 \ B_2 \end{pmatrix} \begin{pmatrix} X \text{ Y} \end{pmatrix} \]

Use path integral formulation (action)

Modes are found by solving the zero eigenvalue equation

\[ (G_{\alpha} - (i\omega_n)^2) \begin{pmatrix} 0 & 1 \\ i \omega_n & 0 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \]

Only 4 are physical.