Physics 220: Problem Set 2 - solution
due April 24, 2012.

1. Show that the longitudinal susceptibility, $\chi_l = \frac{\partial \langle S_x \rangle}{\partial h_\perp}$, of the 1d transverse field Ising chain has a logarithmic divergence at the quantum critical point.

We wrote down the formula for this susceptibility (not sure why I am calling it longitudinal. Probably transverse would be better?) in class (it is the same as the second derivative of the energy):

$$\chi_l = \int_0^\pi \frac{dk}{2\pi} \frac{2J^2 \sin^2 k}{(J^2 + 4h_\perp^2 - 4h_\perp J \cos k)^{3/2}}.$$  \hfill (1)

The critical point occurs when $h_\perp = J/2$. We see that at that precise point, the denominator vanishes when $k = 0$, and the integral becomes divergent. So we let $h_\perp = J/2(1 + t)$, with $|t| \ll 1$, which describes the vicinity of the critical point, and approximate $\cos k \approx 1 - k^2/2$, $\sin k \approx k$. This gives, keeping leading terms

$$\chi_l \approx \frac{1}{\pi J} \int_0^\pi \frac{dk}{2\pi} \frac{k^2}{(t^2 + k^2)^{3/2}}.$$  \hfill (2)

Writing $k = |t|x$, we have

$$\chi_l \approx \frac{1}{\pi J} \int_0^{|t|/t} \frac{dx}{(x^2 + 1)^{3/2}}.$$  \hfill (3)

At small $t$, we see that this is controlled by $x \gg 1$, so can approximate it by

$$\chi_l \approx \frac{1}{\pi J} \int_1^{|t|/t} \frac{dx}{x} \sim \frac{\ln(|t|/|t|)}{\pi J}.$$  \hfill (4)

Technically, this is correct only up to constant terms in $t$, i.e. only the coefficient of the log is correct, and the $\pi$ inside the ln is not reliable. But this is enough for us.

2. For the 3d Ising model (or the 2d quantum transverse field Ising model), the critical point is described by a scale invariant field theory with an “energy density” operator $\varepsilon$ and a spin operator $\sigma$, just as in 2d (1d quantum), but with $d_\varepsilon = 1.59$ and $d_\sigma = 0.52$. Find the specific heat exponent $\alpha$, the order parameter exponent $\beta$, and the correlation length exponent $\nu$.

The effective field theory for the vicinity of the critical point is

$$F = \int d^3x \left\{ \varepsilon - h\sigma \right\},$$  \hfill (5)

where $t \sim (T - T_c)/T_c$ is the deviation from the critical temperature, and $h$ is proportional to the magnetic field. Under a scale transformation, $x \rightarrow bx$, we have $t \rightarrow b^{3-d_\varepsilon} t$ and $h \rightarrow b^{3-d_\sigma} h$.

- To get the specific heat exponent, we need the internal energy density at zero field. This is

$$u = \langle \varepsilon \rangle \sim b^{-d_\varepsilon} f(b^{3-d_\varepsilon} t) \sim |t|^{d_\varepsilon/(3-d_\varepsilon)},$$  \hfill (6)

choosing $b = |t|^{-1/(3-d_\varepsilon)}$. The specific heat is $\partial u/\partial t \sim |t|^{d_\varepsilon/(3-d_\varepsilon) - 1} \equiv |t|^{-\alpha}$, so

$$\alpha = (3 - 2d_\varepsilon)/(3 - d_\varepsilon).$$  \hfill (7)

If you take $d_\varepsilon = 1.59$, you get $\alpha = -0.13$. Actually, I gave the wrong number in the homework. It is really $d_\varepsilon = 1.41$, which gives $\alpha = 0.11$. 
• Again we take zero field, and write
\[ m = \langle \sigma \rangle \sim b^{-d\sigma} f(b^{3-d\varepsilon} t) \sim |t|^{d\sigma/(3-d\varepsilon)}. \] (8)
By definition, \( m \sim |t|^2 \), so
\[ \beta = d\sigma/(3 - d\varepsilon). \] (9)
Using the values given in the homework, we get \( \beta = 0.369 \), but if we use the corrected value, we get \( \beta = 0.328 \).
• Finally, for the correlation length, since it is a length,
\[ \xi \sim b f(b^{3-d\varepsilon} t) \sim |t|^{-1/(3-d\varepsilon)}, \] (10)
which, since \( \xi \sim |t|^{\nu} \) gives
\[ \nu = 1/(3 - d\varepsilon). \] (11)
This gives, using the value from the homework, \( \nu = 0.709 \), or with the correct value, \( \nu = 0.63 \).