Problem 1.

We begin by solving the equations

\[-T_1^2 - T_2^2 + X_1^2 + \ldots + X_d^2 = -L^2,\]  

\[r = T_1 - X_1, \quad t = T_2 L/r, \quad x_i = X_i L/r\]

for the \(T\)'s and \(X\)'s. We have immediately

\[T_2 = \frac{rt}{L}, \quad X_i = \frac{x_i r}{L}.\]

We also have \(X_1 = T_1 - r\). Substitute these into (1) to get

\[-L^2 = -T_1^2 - \frac{r^2 t^2}{L^2} + (T_1 - r)^2 + \sum_{i=2}^d \frac{x_i^2 r^2}{L^2}\]

\[= -2T_1 r + r^2 - \frac{r^2 t^2}{L^2} + \sum_{i=2}^d \frac{x_i^2 r^2}{L^2},\]

which, after solving for \(T_1\), gives

\[T_1 = \frac{L^2}{2r} - \frac{r^2 t^2}{2L^2} + \frac{1}{2} \sum_{i=2}^d \frac{x_i^2 r^2}{L^2} + \frac{r}{2} = f + \frac{r}{2}.\]

Note that \(f = f(t, r, x_i)\). Then

\[X_1 = T_1 - r = f - \frac{r}{2}.\]

The equations for the \(T\)'s and \(X\)'s then imply

\[dT_1 = df + \frac{dr}{2}, \quad dT_2 = \frac{r}{L} dt + \frac{t}{L} dr, \quad dX_1 = df - \frac{dr}{2}, \quad dX_i = \frac{x_i}{L} dr + \frac{r}{L} dx_i.\]

We now substitute these into the flat space metric with signature \((2,d)\). This gives us

\[ds^2 = -dT_1^2 - dT_2^2 + dX_1^2 + \ldots + dX_d^2\]

\[= -\left(df + \frac{dr}{2}\right)^2 - \left(\frac{r}{L} dt + \frac{t}{L} dr\right)^2 + \left(df - \frac{dr}{2}\right)^2 + \sum_{i=2}^d \left(\frac{x_i}{L} dr + \frac{r}{L} dx_i\right)^2\]

\[= -2df dr - \frac{r^2}{L^2} dt^2 - \frac{t^2}{L^2} dr^2 - \frac{2rt}{L^2} dt dr + \sum \left(\frac{x_i^2}{L^2} dr^2 + \frac{r^2}{L^2} dx_i^2 + \frac{2x_i r}{L^2} dx_i dr\right).\]
But from the definition of \( f \),
\[
-2df dr = \frac{L^2}{r^2} dr^2 + \frac{l^2}{L^2} dr^2 + \frac{2tr}{L^2} dtdr - \sum_{i=2}^{d} \frac{2x_i r}{L^2} dx_i dr - \frac{x_i}{L^2} dr^2,
\]
so after all the cancellations,
\[
ds^2 = \frac{r^2}{L^2} (-dt^2 + dx_1^2 + \ldots + dx_d^2) + \frac{L^2}{r^2} dr^2.
\]

**Problem 2.**

Our action is
\[
S = \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} (R + 4(\nabla \phi)^2) - \frac{1}{2(p + 2)!} F_{\mu \nu}^2 \right]. \tag{2}
\]

We wish to make a conformal transformation, \( \tilde{g}_{\mu \nu} = \omega^2(x) g_{\mu \nu} \) such that \( \sqrt{-g} e^{-2\phi} R = \sqrt{-\tilde{g}} \tilde{R} + \ldots \). Since we are working in 10 dimensions,
\[
\sqrt{-g} = \sqrt{-\det(\omega^{-2} \tilde{g}_{\mu \nu})} = \omega^{-10} \sqrt{-\tilde{g}}.
\]
The Ricci scalar transforms as (see for instance Carrol (G.16))
\[
R = \omega^2 \tilde{R} + \ldots,
\]
so
\[
\sqrt{-g} e^{-2\phi} R = \omega^{-8} e^{-2\phi} \sqrt{-\tilde{g}} \tilde{R} + \ldots.
\]
We therefore identify \( \omega = e^{-\phi/4} \) and \( g_{\mu \nu} = e^{\phi/2} \tilde{g}_{\mu \nu} \). We now perform this transformation on our action. The determinant transforms as
\[
\sqrt{-g} = e^{5\phi/2} \sqrt{-\tilde{g}}.
\]
The Ricci scalar transforms according to Carrol (G.16) as
\[
e^{-2\phi} R = e^{-2\phi} \left[ e^{-\phi/2} \tilde{R} + 18 \tilde{g}^{\alpha \beta} e^{-\phi/4} (\tilde{\nabla}_\alpha \tilde{\nabla}_\beta e^{-\phi/4}) - 90 \tilde{g}^{\alpha \beta} \tilde{\nabla}_\alpha (e^{-\phi/4}) (\tilde{\nabla}_\beta e^{-\phi/4}) \right]
\]
\[
= e^{-5\phi/2} \left[ \tilde{R} + \frac{9}{8} \left( \frac{1}{4} (\nabla \phi)^2 - (\nabla \phi)^2 \right) - \frac{45}{8} (\nabla \phi)^2 \right]. \tag{3}
\]
To get to the second line, we differentiated the exponentials and then factored them out.

The scalar term in the action has one power of the inverse metric, so it transforms as
\[
e^{-2\phi} 4(\nabla \phi)^2 = e^{-2\phi} 4\tilde{g}^{\alpha \beta} (\nabla_\alpha \phi) (\nabla_\beta \phi) = e^{-5\phi/2} 4\tilde{g}^{\alpha \beta} \tilde{\nabla}_\alpha \phi \tilde{\nabla}_\beta \phi = e^{-5\phi/2} 4(\nabla \phi)^2. \tag{4}
\]
Note that the covariant derivative of a scalar is equivalent to a partial derivative and so doesn’t change under coordinate transformations.

Similarly, the term with the \((p + 2)\) form has \( p + 2 \) powers of the inverse metric so it transforms as
\[
- \frac{1}{2(p + 2)!} F_{\mu \nu}^2 = - \frac{1}{2(p + 2)!} (e^{-\phi/2})^{p+2} \tilde{F}_{\mu \nu}^2, \tag{5}
\]
where the square in $\tilde{F}_{p+2}^2$ uses the metric $\tilde{g}_{\mu\nu}$. Note that the field tensor $F = dA$ itself does not change under conformal transformations as the exterior derivative is defined using partial derivatives (covariant derivatives of antisymmetric tensors are equivalent to partial derivatives due to the symmetry of the connection coefficients). When we put eqs. (3)–(5) into (2), we get

$$S = \int d^{10}x \sqrt{-\tilde{g}} \left[ \tilde{R} - \frac{1}{2} (\tilde{\nabla} \phi)^2 - \frac{1}{2(p+2)!} \epsilon^{(3-p)(\phi/2)\tilde{F}_{p+2}^2} + \left( -\frac{9}{2} \tilde{\nabla}^2 \phi \right) \right]. \checkmark$$

The last term is a total derivative and can be dropped.

**Problem 3.**

(a) We wish to find the equations of motion from the action (2). We have three fields: $A_{p+1}$, $\phi$, and $g_{MN}$. Let $\mathcal{L}$ be the Lagrangian density (without the $\sqrt{-g}$ factor). Then the equations of motion for $A_{p+1}$ and $\phi$ are given by the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial A_{p+1}} - \nabla_M \frac{\partial \mathcal{L}}{\partial (\nabla_M A_{p+1})} = 0, \quad \frac{\partial \mathcal{L}}{\partial \phi} - \nabla_M \frac{\partial \mathcal{L}}{\partial (\nabla_M \phi)} = 0.$$

We have $\frac{\partial \mathcal{L}}{\partial A_{p+1}} = 0$, so the first equation gives

$$0 = \nabla_M \frac{\partial \mathcal{L}}{\partial (\nabla_M A_{p+1})} \propto \nabla L_1 g^{M_1 N_1} \cdots g^{M_{p+2} N_{p+2}} \frac{\partial}{\partial \nabla L_1 A_{L_2 \cdots L_{p+2}}} F_{M_1 \cdots M_{p+2}} F_{N_1 \cdots N_{p+2}}$$

$$\propto \nabla L_1 g^{M_1 N_1} \cdots g^{M_{p+2} N_{p+2}} F_{M_1 \cdots M_{p+2}} \frac{\partial}{\partial \nabla L_1 A_{L_2 \cdots L_{p+2}}} F_{N_1 \cdots N_{p+2}}$$

$$\propto \nabla L_1 g^{M_1 N_1} \cdots g^{M_{p+2} N_{p+2}} F_{M_1 \cdots M_{p+2}} \frac{\partial}{\partial \nabla L_1 A_{L_2 \cdots L_{p+2}}} (N_1 A_{N_2 \cdots N_{p+2}})$$

$$= \nabla L_1 g^{M_1 N_1} \cdots g^{M_{p+2} N_{p+2}} F_{M_1 \cdots M_{p+2}} \delta L_1 [N_1 \delta L_2 N_2 \cdots \delta L_{p+2} N_{p+2}]$$

$$\propto \nabla L_1 F_{L_1 L_2 \cdots L_{p+2}}. \checkmark$$

(6)

The zero on the LHS lets us ignore all overall factors. To get the third line, we used the product rule and relabeled dummy indices.

Now we compute the scalar equation. We have

$$\frac{\partial \mathcal{L}}{\partial \phi} = -2e^{-2\phi} (R + 4(\nabla \phi)^2).$$

Also,

$$-\nabla M \frac{\partial \mathcal{L}}{\partial \nabla M \phi} = -4\nabla M \left[ e^{-2\phi} g^{RS} \frac{\partial}{\partial \nabla M \phi} \nabla R \phi \nabla S \phi \right]$$

$$= -4\nabla M \left[ e^{-2\phi} g^{RS} (\delta^M_R \nabla S \phi + \nabla R \phi \delta^M_S) \right]$$

$$= -8\nabla M \left( e^{-2\phi} \nabla M \phi \right)$$

$$= -8e^{-2\phi} (\nabla^2 \phi - 2(\nabla \phi)^2).$$
Putting these into the Euler-Lagrange equations, we get
\[ \nabla^2 \phi - (\nabla \phi)^2 + \frac{1}{4} R = 0. \] (7)

Now we derive Einstein’s equations. Write the action as
\[ S = \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} R + \mathcal{L}_m \right]. \]

The term \( \mathcal{L}_m \) will give us the stress tensor \( T_{MN} \). To find the contribution from \( e^{-2\phi} R \), we will follow Carroll pp. 161-163 with an extra factor of \( e^{-2\phi} \) where appropriate. The variation of \( S_H \) consists of three terms (Carroll 4.57)
\[ \delta S_H = (\delta S_1) + (\delta S_2) + (\delta S_3), \]

which each term is given by Carroll (4.58, 4.65, 4.69)
\[ (\delta S_1) = \int d^{10}x \sqrt{-g} e^{-2\phi} g^{MN} \delta R_{MN} = \int d^{10}x \sqrt{-g} e^{-2\phi} \nabla_S \left[ g_{MN} \nabla^S (\delta g^{MN}) - \nabla_L (\delta g^{SL}) \right] \]
\[ (\delta S_2) = \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} R_{MN} \right] \delta g^{MN} \]
\[ (\delta S_3) = \int d^{10}x \sqrt{-g} e^{-2\phi} R \delta \sqrt{-g} = \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} \left( -\frac{1}{2} g_{MN} R \right) \right] \delta g^{MN}. \]

In the usual Einstein-Hilbert action, the term \( (\delta S_1) \) vanishes, but we have an extra \( e^{-2\phi} \) factor. Integrating \( (\delta S_1) \) by parts twice gives us
\[ (\delta S_1) = \int d^{10}x \sqrt{-g} \left[ 2 e^{-2\phi} \nabla_S \phi \right] g_{MN} \nabla^S (\delta g^{MN}) - \nabla_L (\delta g^{SL}) \]
\[ = \int d^{10}x \sqrt{-g} \left[ -\nabla_S \left( 2 g_{MN} e^{-2\phi} \nabla_S \phi \right) + \nabla_N \left( 2 e^{-2\phi} \nabla_M \phi \right) \right] \delta g^{MN} \]
\[ = \int d^{10}x \sqrt{-g} e^{-2\phi} \left[ 2 \nabla_M \nabla_N \phi - 4 \nabla_M \phi \nabla_N \phi - 2 g_{MN} \nabla^2 \phi + 4 g_{MN} (\nabla \phi)^2 \right] \delta g^{MN}. \]

Then putting all of the \( \delta S \)’s together, Einstein’s equation becomes
\[ e^{-2\phi} \left[ R_{MN} - \frac{1}{2} g_{MN} R + 2 \nabla_M \nabla_N \phi - 4 \nabla_M \phi \nabla_N \phi - 2 g_{MN} \nabla^2 \phi + 4 g_{MN} (\nabla \phi)^2 \right] = T_{MN}. \] (8)

Here, \( T_{MN} \) is the stress tensor defined by
\[ T_{MN} = -\frac{1}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{MN}} = -\frac{\partial \mathcal{L}_m}{\partial g^{MN}} + \frac{1}{2} g_{MN} \mathcal{L}_m, \] (9)

where in the last step we used Carroll (4.69). But \( \mathcal{L}_m \) is given by
\[ \mathcal{L}_m = 4 e^{-2\phi} (\nabla \phi)^2 - \frac{1}{2(p + 2)!} F^2, \] (10)
Ansatz and notation

The nontrivial components of the index $L$ Field Tensor equation components of the indices $M$ and show that the equations of motion (6), (7), and (12) are equivalent to Laplace’s equation.

First, some bookkeeping. Capital indices (e.g. $M$ and lower case indices (e.g. $a, i, j$) run from $p + 1, \ldots, 9$. Using these indices, the metric ansatz above will then be represented as

$$g_{MN} = H^{-1/2} \eta_{\mu \nu} dx^\mu dx^\nu + H(y)^{1/2} dy_i dy^i.$$

and show that the equations of motion (6), (7), and (12) are equivalent to Laplace’s equation.

First, some bookkeeping. Capital indices (e.g. $M, N$) run from $0, \ldots, 9$, greek indices (e.g. $\mu, \nu$) run from $0, \ldots, p$, and lower case indices (e.g. $a, i, j$) run from $p + 1, \ldots, 9$. Using these indices, the metric ansatz above will then be represented as

$$g_{MN} = H^{-1/2} \eta_{\mu \nu} + H^{1/2} \delta_{ij}.$$

Also, we define squares involving partial derivatives as contractions with $\delta_{ij}$. Thus,

$$\partial^2 H \equiv \delta_{ij} \partial_i \partial_j H, \quad (\partial H)^2 \equiv \delta_{ij} \partial_i H \partial_j H.$$

**Field Tensor equation** – Let us now consider the equation for the field tensor (6) under this ansatz.

$$0 = \nabla L F^{LM_0 \ldots M_p} \propto \nabla L \nabla ^L A^{M_0 \ldots M_p}.$$

The nontrivial components of the index $L$ must be in $p + 1, \ldots, 9$ since $A = A(y)$. The components of the indices $M_i$ must be in $0, \ldots, p$ by the ansatz for $A$. Therefore,

$$0 = \nabla_i \nabla ^i A^{\mu_0 \ldots \mu_p} \propto \nabla_i \nabla ^i A^{[\mu_0 \ldots \mu_p]} = \nabla_i \nabla ^i A^{\mu_0 \ldots \mu_p}.$$
By the antisymmetry of $A$ and the ansatz, this reduces to

$$0 = \nabla_i \nabla^i A^{0\ldots p}$$

$$= \frac{1}{\sqrt{-g}} \partial_k \left( \sqrt{-g} g^{00} \ldots g^{pp} \partial_j (A_{0\ldots p}) \right)$$

$$\propto \partial_k \left( \sqrt{-g} (H^{1/2})^{p+1} H^{-1/2} \delta^i_j \left( -H^{-2} \partial_j H \right) \right)$$

$$= \delta^i_j \partial_k \left( \sqrt{-g} H^{(p+1)/2} \partial_j H \right)$$

$$= \delta^i_j \partial_k \partial_j H$$

$$= \partial^2 H . \sqrt{\gamma}$$

Therefore, (6) reduces to Laplace’s equations. We have used, and will use often again, the expression for the determinant $\sqrt{-g} = H^{(4-p)/2}$.

**Ricci Tensor and Scalar** – There are curvature terms in the equation for the dilaton and Einstein’s equations. Here we will compute $R_{MN}$ by conformal transformations and then contract to get $R$. Let

$$\tilde{d}s^2 = H^{1/2} ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + H dy_i dy^i,$$

with associated Ricci tensor $\tilde{R}_{MN}$. We label the second term in that metric with the line element $d\ell^2$ with Ricci tensor $R_{ij}^{(f)}$. Since $\eta_{\mu\nu}$ is Ricci flat, we have

$$\tilde{R}_{MN} = R_{ij}^{(f)} .$$

If we rescale $d\ell^2$, we can recover flat Euclidean space:

$$\tilde{d}\ell^2 = H^{-1} d\ell^2 = dy_i dy^i, \quad \tilde{R}_{ij}^{(f)} = 0 .$$

Then using Carroll (G.15) with $\omega = H^{-1/2}$ and $\tilde{g}_{ij} = \delta^i_j$, we find

$$\tilde{R}_{MN} = R_{ij}^{(f)} = \tilde{R}_{ij}^{(f)} + \left[ (7 - p) \delta^k_i \delta^j_l + \delta_{ij} \delta^{kl} \right] H^{1/2} (\partial_k \partial_l H^{-1/2})$$

$$- (8 - p) \delta_{ij} \delta^{kl} H (\partial_k H^{-1/2}) (\partial_l H^{-1/2})$$

$$= H^{1/2} \left[ (7 - p) \partial_i \partial_j H^{-1/2} + \delta_{ij} \partial^2 H^{-1/2} \right] + (p - 8) \delta_{ij} H \left( \partial H^{-1/2} \right)^2$$

$$= \frac{p - 7}{2} H^{-2} \left( H \partial_i \partial_j H - \frac{3}{2} \partial_i H \partial_j H \right) + \frac{1}{4} \delta_{ij} H^{-2} \left[ -2 H \partial^2 H + (p - 5) (\partial H)^2 \right] .$$

(13)

Now that we have $\tilde{R}_{MN}$, we can use Carroll (G.15) again to find $R_{MN}$. This time, we have $\omega = H^{1/4}$ and $\tilde{g}_{MN} = \eta_{\mu\nu} + H \delta_{ij}$,

$$R_{MN} = \tilde{R}_{MN} + \left[ 8 \delta_M^A \delta_N^B + \tilde{g}_{MN} \tilde{g}^{AB} \right] H^{-1/4} (\nabla_A \nabla_B H^{1/4}) - 9 \tilde{g}_{MN} \tilde{g}^{AB} H^{-1/2} (\nabla_A H^{1/4}) (\nabla_B H^{1/4})$$

$$= \tilde{R}_{MN} + 8 H^{1/4} \nabla_M \nabla_N H^{-1/4} + \tilde{g}_{MN} \tilde{g}^{AB} H^{-1/4} (\nabla_A \nabla_B H^{1/4})$$

$$- 9 \tilde{g}_{MN} \tilde{g}^{AB} H^{-1/2} (\nabla_A H^{1/4}) (\nabla_B H^{1/4}) .$$

(14)
We now compute (14) term by term. We will need the Christoffel symbol

\[ \tilde{\Gamma}_{MN}^K = \frac{1}{2} \tilde{g}^{KA} (\partial_M \tilde{g}_{NA} + \partial_N \tilde{g}_{AM} - \partial_A \tilde{g}_{MN}) \]
\[ = \frac{1}{2} H^{-1} \delta^{ka} (\delta_{ja} \partial_i H + \delta_{ai} \partial_j H - \delta_{ij} \partial_a H) \]
\[ = \frac{1}{2} H^{-1} \left( \delta^k_j \partial_i H + \delta^k_i \partial_j H + \delta_{ij} \delta^{ka} \partial_a H \right) . \]

Then
\[ 8H^{-1/4} \tilde{\nabla}_M \tilde{\nabla}_N H^{1/4} = 8H^{-1/4} \nabla_M \left( \frac{1}{4} H^{-3/4} \partial_N H \right) \]
\[ = 8H^{-1/4} \left[ \partial_M \left( \frac{1}{4} H^{-3/4} \partial_N H \right) - \tilde{\Gamma}_{MN}^K \left( \frac{1}{4} H^{-3/4} \partial_K H \right) \right] \]
\[ = H^{-2} \left[ 2H \partial_i \partial_j H - \frac{7}{2} \partial_i \partial_j H + \delta_{ij} (\partial H)^2 \right] . \quad (15) \]

Continuing with the remaining terms, we have
\[ \tilde{g}^{MN} \tilde{g}^{AB} H^{-1/4} \left( \nabla_A \nabla_B H^{1/4} \right) = \tilde{g}^{MN} H^{-1/4} \frac{1}{\sqrt{-g}} \partial_A \left( \sqrt{-g} \tilde{g}^{AB} \partial_B H^{1/4} \right) \]
\[ = \tilde{g}^{MN} H^{-1/4} H^{(9-p)/2} \delta^i_j \left( H^{(9-p)/2} H^{-1} \delta^{ij} \frac{1}{4} H^{-3/4} \partial_j H \right) \]
\[ = \frac{1}{4} \tilde{g}^{MN} H^{-3} \left( H \partial^2 H + \frac{11 - 2p}{4} (\partial H)^2 \right) \]
\[ = \frac{1}{4} (\eta_{\mu \nu} + H \delta_{ij}) H^{-3} \left( H \partial^2 H + \frac{11 - 2p}{4} (\partial H)^2 \right) , \quad (16) \]

and
\[ -9 \tilde{g}^{MN} \tilde{g}^{AB} H^{-1/2} (\nabla_A H^{1/4}) (\nabla_B H^{1/4}) = -9 \tilde{g}^{MN} H^{-3/2} \left( \frac{1}{4} H^{-3/4} \right)^2 \delta_{ij} \partial_i H \partial_j H \]
\[ = -\frac{9}{16} (\eta_{\mu \nu} + H \delta_{ij}) H^{-3} (\partial H)^2 . \quad (17) \]

Finally, putting (13), (15), (16), (17) into (14) and combining like terms, we get
\[ R_{MN} = H^{-2} \left[ \frac{p - 3}{2} H \partial_i \partial_j H + \frac{7 - 3p}{4} \partial_i H \partial_j H - \frac{1}{4} \delta_{ij} H \partial^2 H + \frac{p - 1}{8} \delta_{ij} (\partial H)^2 \right] \]
\[ + \eta_{\mu \nu} H^{-3} \left[ \frac{1}{4} H \partial^2 H + \frac{1 - p}{8} (\partial H)^2 \right] . \quad (18) \]

Contracting this with $g^{MN}$, we get
\[ R = g^{MN} R_{MN} = H^{-5/2} \left[ \frac{2p - 7}{2} H \partial^2 H + \frac{(1 + p)(3 - p)}{4} (\partial H)^2 \right] \quad (19) \]

**Dilaton equation** – The dilaton equation (7) is equivalent to
\[ R + 4 \nabla^2 \phi - 4 (\nabla \phi)^2 = 0 . \quad (20) \]
The second term is given by
\[
4 \nabla^2 \phi = \frac{4}{\sqrt{-g}} \nabla_M \left( \sqrt{-g} g^{MN} \nabla_N \phi \right)
\]
\[
= 4H^{(p-4)/2} \nabla_M \left( H^{(1-p)/2} g^{MN} \nabla_N \left( \frac{3-p}{4} \ln H \right) \right)
\]
\[
= (3-p)H^{(p-4)/2} \partial_i \left( H^{(1-p)/2} \delta^{ij} \partial_j H \right)
\]
\[
= (3-p)H^{-5/2} \left[ H \partial^2 H + \frac{1-p}{2} (\partial H)^2 \right],
\]
(21)
while the third term is given by
\[
-4(\nabla \phi)^2 = -4H^{-1/2} \delta^{ij} \partial_i \left( \frac{3-p}{4} \ln H \right) \partial_j \left( \frac{3-p}{4} \ln H \right) = -\frac{(3-p)^2}{4} H^{-5/2} (\partial H)^2 \quad (22)
\]

Putting (19), (21), and (22) into (20) we see that most terms cancel out and we are left with
\[
-\frac{1}{2} H^{-3/2} \partial^2 H = 0 \iff \partial^2 H = 0.
\]
so the dilaton equation reduces to Laplace’s equation.

**Einstein’s equations** – It is convenient to rewrite Einstein’s equations (12) as
\[
R_{MN} - \frac{1}{2} g_{MN} \left( R + 4 \nabla^2 \phi - 4(\nabla \phi)^2 \right) = -2 \nabla_M \nabla_N \phi - \frac{1}{4(p+2)!} g_{MN} e^{2\phi} F^2 + \frac{1}{2(p+1)!} e^{2\phi} F_M^2.
\]
(23)

The second term on the LHS vanishes by the equations of motion for the dilaton (20). Therefore, we only need to compute the RHS and compare it to the Ricci tensor given by (18). We will do this term by term. The first term involves the Christoffel symbols
\[
\Gamma^K_{MN} = \frac{1}{2} g^{KL} \left( \partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN} \right)
\]
\[
= \frac{1}{2} g^{kl} \left( \partial_k g_{jl} + \partial_j g_{kl} - \partial_l g_{jk} \right)
\]
\[
= \frac{1}{4} H^{-2} \delta^{kl} \left( \delta_{jl} H \partial_k H + \delta_{kl} H \partial_j H - \delta_{ij} H \partial_k H + \eta_{\mu \nu} \partial_k H \right).
\]

Then the first term becomes
\[
-2 \nabla_M \nabla_N \phi = \frac{p-3}{2} \left[ \partial_M (H^{-1} \partial_N H) - \Gamma^K_{MN} (H^{-1} \partial_K H) \right]
\]
\[
= \frac{p-3}{2} H^{-2} \left[ H \partial_i \partial_j H - \frac{3}{2} \partial_i H \partial_j H + \frac{1}{4} \delta_{ij} (\partial H)^2 \right] + \frac{3-p}{8} \eta_{\mu \nu} H^{-3} (\partial H)^2.
\]
(24)
Now for the $F^2$ terms. Note that $\partial_L A_{K_0 \ldots K_p}$ is only nontrivial when $L \in \{p+1, \ldots, 9\}$ and $\{K_0, \ldots, K_p\} = \{0, \ldots, p\}$. Therefore, $F_{M_1 \ldots M_{p+2}}$ is only nontrivial if $\{M_1, \ldots, M_{p+2}\} =$
\{0, \ldots, p, i\} for some \(i \in \{p + 1, \ldots, 9\}\). From this, we can conclude that \(F_{\mu i}^2 = 0\), and \(F_{\mu \nu}^2\) must be diagonal. Now let’s compute \(F_{ij}^2\) by summing over permutations:

\[
F_{ij}^2 = F_{i\mu_0 \ldots \mu_p} F_{j}^{\mu_0 \ldots \mu_p} \\
= (p + 1)! F_{i0 \ldots p} F_{j}^{0 \ldots p} \\
= (p + 1)! (p + 2)^2 \partial_i A_{0 \ldots p} \partial_j A_{0 \ldots p} g^{00} \ldots g^{pp} \\
= (p + 1)! \left(\frac{(p + 2)^2}{(p + 2)!}\right)^2 \left(\sum_{\text{perm. of } (0, \ldots, p)} \partial_i A_{0 \ldots p}\right) \left(\sum_{\text{perm}} \partial_j A_{0 \ldots p}\right) g^{00} \ldots g^{pp} \\
= (p + 1)! (\partial_i H^{-1})(\partial_j H^{-1}) g^{00} \ldots g^{pp} \\
= -(p + 1)! H^{(p-7)/2} \partial_i H \partial_j H . \tag{25}
\]

Now for \(F_{\mu \nu}^2\). We have

\[
F_{\mu \nu}^2 = (p + 1)! F_{\mu \tilde{\mu} \ldots \tilde{\mu} i} F_{\nu 0 \ldots \tilde{\mu} \ldots \tilde{\mu} j} ,
\]

where the \(\tilde{\mu}\) means the \(\mu\)-th element is missing in the series \(0 \ldots p\). Since \(F_{\mu \nu}^2\) must be diagonal, we can replace this with

\[
F_{\mu \nu}^2 = (p + 1)! \sum_i F_{\mu 0 \ldots \tilde{\mu} \ldots \tilde{\mu} i} F_{\mu 0 \ldots \tilde{\mu} \ldots \tilde{\mu} i} g_{\mu \nu} \quad \text{(no sum)}
\]

\[
= (p + 1)! \eta_{\mu \nu} H^{-1/2} \sum_i F_{i0 \ldots p} F_{i0 \ldots p} \\
= (p + 1)! \eta_{\mu \nu} H^{-1/2} \delta_{ij} F_{i0 \ldots p} F_{j0 \ldots p} \\
= -(p + 1)! \eta_{\mu \nu} H^{(p-9)/2} (\partial H)^2 ,
\]

where in the last line we just follow (25) with an extra factor of \(\eta_{\mu \nu} H^{-1} \delta_{ij}\). Then putting these together, we get

\[
F_{MN}^2 = F_{\mu \nu}^2 + F_{ij}^2 = -(p + 1)! H^{(p-7)/2} \left(\partial_i H \partial_j H + \eta_{\mu \nu} H^{-1} (\partial H)^2\right) .
\]

Then this implies

\[
-\frac{1}{2(p + 1)!} e^{2\phi} F_{MN}^2 = -\frac{1}{2} H^{-2} \left[\partial_i H \partial_j H + \eta_{\mu \nu} H^{-1} (\partial H)^2\right]
\]

and

\[
-\frac{1}{4(p + 2)!} e^{2\phi} g_{MN} F^2 = \frac{1}{4} H^{-2} (\delta_{ij} + H^{-1} \eta_{\mu \nu}) (\partial H)^2
\]

Combining these two with (24), the RHS of (23) is

\[
H^{-2} \left[\frac{p - 3}{2} H \partial_i \partial_j H + \frac{7 - 3p}{4} \partial_i H \partial_j H + \frac{p - 1}{8} \delta_{ij} (\partial H)^2\right] + \eta_{\mu \nu} H^{-3} \left[\frac{1 - p}{8} (\partial H)^2\right] .
\]

Comparing this with the Ricci tensor (18), we see that all terms not proportional to \(\partial^2 H\) cancel out, so we are left again with Laplace’s equation. ✓