Homework 2, due May 23

1. Consider the equation 7.2.4 of Polchinski:

\[ \langle : \exp(ik_1X(z_1, \bar{z}_1)) : \cdots : \exp(ikX(z_n, \bar{z}_n)) : \rangle_{T^2} = iZ_{T^2}(\tau)(2\pi)^d\delta^d(\sum k_i) \] (1)

\[ \times \prod_{i<j} \left| \frac{2\pi}{\partial_\nu \theta_{1}(\nu_1;\tau)} \right|_{\nu=0} \theta_{1} \left( \frac{w_{ij}}{2\pi}; \tau \right) \exp \left[ -\frac{\Im m(w_{ij})^2}{4\pi \Im m(\tau)} \right] \] (2)

To build an amplitude of the closed bosonic string, we would integrate this expression over the worldsheet for all vertex operators, by using \( \prod \int d^2z_i \). Moreover, we would require that the vertex operators are on-shell (their conformal weights are (1, 1)) and we would multiply it by the \( b, c \) ghost integral and by the modular parameter integration.

Show that such an amplitude is modular invariant.

2. **The size of a string.** Consider a closed string at large level \( N \). For a typical state in this ensemble, we would like to know how big does the string grow, that is, we would like to know the average size:

\[ \langle \delta X_1^2 \rangle \] (3)

where \( \delta X \) measures the deviation from the center of mass, in the light-cone.

(a) Use equipartition to state how much energy is stored in the \( X_1 \) oscillators (do this both for the bosonic and the fermionic string).

(b) Show that the quantity \( \delta X_1^2 \) can be decomposed into a sum over the modes of the string.

(c) Evaluate the classical value of \( \delta X_1^2 \) by using equipartition over the modes: assume it is characterized by a typical temperature \( T \). Do you get a convergent integral? Notice that this is not the same question as to how much energy is stored in each mode which would be clearly divergent.

(d) Let us now consider the size of the string in the ground state. How big is the string if we take into account the zero point motion? If the answer is infinite, we should cut it at some resolution, so that only the zero point motion of modes below some fixed cutoff (a physical scale that depends on an experiment) is accessible. If a cutoff is necessary, how does the size grow with the cutoff?
3. Consider a brane (called brane A) that extends in the $0, \ldots, p$ dimensions, and another brane (called brane B) that extends in the $0, 1, \ldots, p, p+q$ directions in the bosonic string. Assume that the branes are separated by some distance $d$ (let us say along the $p+q+1$ direction), and assume that we have a string with a left end at $A$ and a right end at $B$.

(a) What are the boundary conditions for $X^0, \ldots, X^{25}$ on the worldsheet (consider both left and right boundary conditions).

(b) What is the zero point energy of the string? Remember to include the separation of the branes.

*Hint:* There are two ways to do this: either you compute the mode expansion for the modes (let us say in lightcone gauge) and calculate the zero point energy this way, or you can compute the expectation value

$$\langle T(z) \rangle$$

Remember $T(z)$ is normal ordered to remove the singularity, but the finite parts matter. This computation can be done by the method of images by taking a limit

$$\lim_{\epsilon \to 0} \langle : \partial X(z) \partial X(z + \epsilon) : \rangle$$

after removing the singular part, where the Green’s function is computed by placing images of $z$ at appropriate places with various signs, plus the classical contribution from the background $X$. Both computations should give the same result.

(c) Calculate the cylinder amplitude for these open strings and write it in terms of $\theta$-functions and $\eta$-functions, so that one can do an easy modular transformation on it. Remember to include the integral over the momentum of the string.
(d) Write the integral in the closed string channel, and focus on the
contribution from massless string states. Part of this should be
the gravitational potential between the branes. Do you get the
right fall-off? (Hint: the right fall-off should be associated to the
potential of the larger brane, you should justify this.)

4. Some properties of the Weierstrass $\wp$-function.

(a) Let $\wp(z, \tau)$ be the Weierstrass $\wp$-function as described in the lec-
ture notes, and with Laurent expansion coefficients around the
origin given by

$$\wp(z; \tau) = \frac{1}{z^2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + \ldots$$  \hspace{1cm} (6)

Show that $\wp$ satisfies the differential equation

$$(\wp(z)')^2 = 4 \wp(z)^3 - g_2 \wp(z) - g_3$$ \hspace{1cm} (7)

where the derivatives are taken with respect to $z$.

(b) The Eisenstein series of weight 2 $k$ (for $k \geq 2$) is given by

$$G_{2k}(\tau) = \sum_{n,m} \Omega_{m,n}^{-2k}$$ \hspace{1cm} (8)

where $\Omega_{m,n} = m + n \tau$, and the sum is over pairs of integers $(m, n)$
for which $\Omega_{m,n} \neq 0$. The restriction on $k$ is so that the series is
absolutely convergent. Show that

$$G_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} G_{2k}(\tau)$$ \hspace{1cm} (9)

for a fractional modular transformation of $\tau$. In this expression
$a, b, c, d$ are integers and $ad - bc = 1$.

(c) Consider the function $f(z, \tau)$ of $z, \tau$ given by the following expres-
sion

$$f(z, \tau) = \partial_z^2 \log(\theta_1(z; \tau))$$ \hspace{1cm} (10)

Show that $f(z, \tau) = \wp(z, \tau) + c$, where $c$ is a constant. (Hint:
Show that $f$ is doubly periodic in $z$, and that it has a double pole
at zero and no other poles.)