Assignment 4 Solutions

1. Here are the Penrose diagrams

(a) Particle with charge $+q$

(b) Particle with charge $-q$

(c)
2. The metric of the hypersurface \( t = \text{constant}, \ r = r_+ = M + \sqrt{M^2 - a^2} \) is given by setting \( dt = dr = 0 \) and \( r = r_+ \) in the Kerr metric. This gives

\[
ds^2 = \Sigma \, d\theta^2 + \frac{(r_+^2 + a^2)^2 \sin^2 \theta}{\Sigma} \, d\phi^2, \tag{1}\]

where we have defined

\[
\Sigma = \Sigma(r_+) = r_+^2 + a^2 \cos^2 \theta. \tag{2}\]

After some calculation (e.g. using Mathematica), we find that the Ricci scalar of this metric is

\[
R = \frac{2(a^2 + r_+^2)(r_+^2 - 3a^2 \cos^2 \theta)}{\Sigma^3} \tag{3}
= \frac{(-2)(r_+^2 + a^2)}{\sin \theta} \partial_\theta \left( \frac{\cos \theta}{\Sigma^2} \right). \tag{4}
\]

This means \( R < 0 \) when

\[
r_+^2 - 3a^2 \cos^2 \theta < 0. \tag{5}\]

For \( \theta = 0 \), this expression vanishes for \( a = r_+ / \sqrt{3} \). If \( a \) is larger than this critical value, there will be a range of \( \theta \) for which (5) holds. To see this, let \( \sqrt{3}a/r_+ = \varepsilon \) for some \( \varepsilon > 1 \) so that (5) becomes \( \cos^2 \theta > 1/\varepsilon \), which has a solution for some range of \( \theta \). The critical value \( a = r_+ / \sqrt{3} \) corresponds to

\[
r_+^2 - 3a^2 = (M + \sqrt{M^2 - a^2})^2 - 3a^2 = 0. \tag{6}\]

This has two solutions: \( a = 0 \) (we can ignore this case) and \( a = \sqrt{3}M/2 \). Thus for \( a > \sqrt{3}M/2 \) there is a range of \( \theta \) for which (5) holds and for which \( R < 0 \).

Using the Gauss-Bonnet Theorem, the Euler number \( \chi \) is

\[
\chi = \frac{1}{4\pi} \int d\phi \int d\theta \sqrt{g} R
= \frac{1}{4\pi} (2\pi) \int_0^{\pi} d\theta (r_+^2 + a^2) \sin \theta \left[ \frac{(-2)(r_+^2 + a^2)}{\sin \theta} \partial_\theta \left( \frac{\cos \theta}{\Sigma^2} \right) \right]
= -(r_+^2 + a^2)^2 \left( \frac{\cos \theta}{\Sigma^2} \right) \bigg|_0^\pi
= (r_+^2 + a^2)^2 \left[ \frac{1}{(r_+^2 + a^2)^2} + \frac{1}{(r_+^2 + a^2)^2} \right]
= 2
\]

so the horizon is topologically a two sphere.
3. Carroll, problem 6.2

(a) Here we will use Carroll’s notation:

\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \Delta = r^2 - 2GMr + a^2. \]  

We will take \( \theta = \pi/2 \) from now on, so \( \rho = r \). Setting \( d\theta = 0 \) and \( \theta = \pi/2 \) in the Kerr metric gives

\[ ds^2 = - \left( 1 - \frac{2GM}{r} \right) dt^2 - \frac{4GMA}{r} dt d\phi + \frac{\Delta}{\Delta} dr^2 + \frac{\Sigma^2}{r^2} d\phi^2. \]  

A photon orbit \( x^a(\lambda) \) in the equatorial plane \( \theta = \pi/2 \) has tangent vector

\[ P^a = \frac{dx^a}{d\lambda} = \left( \frac{dt}{d\lambda}, \frac{dr}{d\lambda}, 0, \frac{d\phi}{d\lambda} \right). \]  

The Killing vectors \( \xi^a = (\partial_t)^a = \delta_t^a \) and \( \psi^a = (\partial_\phi)^a = \delta_\phi^a \) give two conserved quantities,

\[ E = -P_\xi = \left( 1 - \frac{2GM}{r} \right) \frac{dt}{d\lambda} + \left( \frac{2GMA}{r} \right) \frac{d\phi}{d\lambda}, \]  

\[ L = P_\psi = - \left( \frac{2GMA}{r} \right) \frac{dt}{d\lambda} + \left( \frac{\Sigma^2}{r^2} \right) \frac{d\phi}{d\lambda}. \]

In addition, we have the null condition,

\[ 0 = - \left( 1 - \frac{2GM}{r} \right) \left( \frac{dt}{d\lambda} \right)^2 - \frac{4GMA}{r} \left( \frac{dt}{d\lambda} \right) \left( \frac{d\phi}{d\lambda} \right) + \frac{\Delta}{\Delta} \left( \frac{dr}{d\lambda} \right)^2 + \frac{\Sigma^2}{r^2} \left( \frac{d\phi}{d\lambda} \right)^2. \]

Solving (10) and (11) for \( dt/d\lambda \) and \( d\phi/d\lambda \) and substituting them into the null condition gives (after a bit of algebra)

\[ \left( \frac{dr}{d\lambda} \right)^2 = -\frac{L^2}{r^2} + E^2 \left( 1 + \frac{a^2}{r^2} \right) + \frac{2ML^2}{r^3}. \]  

We want to write this as

\[ \left( \frac{dr}{d\lambda} \right)^2 = \frac{\Sigma^2}{r^4} (E - LW_+)(E - LW_-) \]

\[ = \frac{\Sigma^2}{r^4} \left[ E^2 + L^2W_+W_- - EL(W_+ + W_-) \right]. \]

Comparing the coefficients of the last line with the result (12) gives two equations for \( W_\pm \) and solving these gives

\[ W_\pm = \frac{r^2}{\Sigma^2} \left( \frac{2GMA}{r} \pm \sqrt{\Delta} \right). \]

(b) If \( \Sigma^2 > 0 \), then in order for a turning point \( (dr/d\lambda = 0) \) to exist we would need \( E = LW_+ \). Consider the term \( \sqrt{\Delta} \) in \( W_\pm \). We note that \( \Delta = 0 \) at \( r = r_\pm \) and \( \Delta < 0 \) for \( r_- < r < r_+ \), while \( \Delta > 0 \) otherwise. Thus, for \( r_- < r < r_+ \), we see that \( W_\pm \) is complex, so for \( r_- < r < r_+ \) it’s not possible to have \( E = LW_\pm \) (since \( E \) and \( L \) are real). Hence there are no turning points for \( r_- < r < r_+ \).