The Charge Radius of the Proton

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Based on Richard J. Hill, GP:

PRD 82 113005 (2010) [arXiv:1008.4619]

[arXiv:1103.4617]
Matrix element of EM current between nucleon states give rise to two form factors \((q = p_f - p_i)\)

\[
\langle N(p_f)| \sum_q e_q \bar{q} \gamma^\mu q | N(p_i) \rangle = \bar{u}(p_f) \left[ \gamma^\mu F_1^N(q^2) + \frac{i\sigma_{\mu\nu}}{2m} F_2^N(q^2) q^\nu \right] u(p_i)
\]
Form Factors

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\]

- Sachs electric and magnetic form factors \( (t = q^2 = -Q^2) \)

\[
G_E^N(t) = F_1^N(t) + \frac{t}{4m_N^2} F_2^N(t), \quad G_M^N(t) = F_1^N(t) + F_2^N(t).
\]

\[
G_E^P(0) = 1, \quad G_E^N(0) = 0, \quad G_M^P(0) = \mu_p \approx 2.793, \quad G_M^N(0) = \mu_n \approx -1.913
\]
Form Factors

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\langle N(p_f) | \sum_q e_q \bar{q} \gamma^\mu q | N(p_i) \rangle = \bar{u}(p_f) \left[ \gamma_\mu F_1^N(q^2) + \frac{i \sigma_{\mu\nu} F_2^N(q^2) q_\nu}{2m} \right] u(p_i)
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$$

- The slope of $G_E^P$

$$
\langle r^2 \rangle_E^P = 6 \frac{dG_E^P}{dq^2} \bigg|_{q^2=0} \quad \text{or} \quad G_E^P(q^2) = 1 + \frac{q^2}{6} \langle r^2 \rangle_E^P + \ldots,
$$

determines the charge radius $r_E^P \equiv \sqrt{\langle r^2 \rangle_E^P}$
Charge radius from atomic physics

- $G_E^P(t)$ and $G_M^P(t)$: input for precision QED observables for bound proton lepton systems

$$\langle p(p_f) | \sum_q e_q \bar{q} \gamma^\mu q | p(p_i) \rangle = \bar{u}(p_f) \left[ \gamma^\mu F_1^P(q^2) + \frac{i\sigma_{\mu\nu}}{2m} F_2^P(q^2) q^\nu \right] u(p_i)$$
Charge radius from atomic physics

- $G^p_E(t)$ and $G^p_M(t)$: input for precision QED observables for bound proton lepton systems

$$\langle p(p_f) | \sum_q e_q \bar{q} \gamma^\mu q | p(p_i) \rangle = \bar{u}(p_f) \left[ \gamma_\mu F^p_1(q^2) + \frac{i \sigma_{\mu\nu}}{2m} F^p_2(q^2) q_\nu \right] u(p_i)$$

- For a charged point particle: $F_1(0) = 1$ and $F_2(0) = 0$

Amplitude for $p + \ell \rightarrow p + \ell$

$$i\mathcal{M} \approx \frac{ie_\ell e_p}{q^2} \chi^\dagger_p \chi_p \chi^\dagger_\ell \chi_\ell \quad \Rightarrow \quad U(r) = -Z\alpha/r$$
Charge radius from atomic physics

- \( G_E^p(t) \) and \( G_M^p(t) \): input for precision QED observables for bound proton lepton systems

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\langle p(p_f) | \sum_q e_q \bar{q} \gamma^\mu q | p(p_i) \rangle = \bar{u}(p_f) \left[ \gamma_\mu F_1^p(q^2) + \frac{i\sigma_{\mu\nu}}{2m} F_2^p(q^2) q_\nu \right] u(p_i)
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i\mathcal{M} \approx \frac{ie_\ell e_p}{q^2} \chi_p^\dagger \chi_p \chi_\ell^\dagger \chi_\ell \Rightarrow U(r) = -Z\alpha/r
\]

- Including \( q^2 \) corrections from proton

\[
i\mathcal{M} \approx \frac{ie_\ell e_p}{q^2} q^2 \left[ \frac{F_1^p(0)}{8m_p^2} + \frac{dF_1^p}{dq^2} \bigg|_{q^2=0} + \frac{F_2^p(0)}{4m_p^2} \right] \chi_p^\dagger \chi_p \chi_\ell^\dagger \chi_\ell
\]

- Proton structure corrections

\[
U(r) = 4\pi Z\alpha \delta^3(r) \left( \frac{dF_1^p}{dq^2} \bigg|_{q^2=0} + \frac{F_2^p(0)}{4m_p^2} \right) = \frac{4\pi Z\alpha}{6} \delta^3(r)(r_E^p)^2
\]
Charge radius from atomic physics

- Proton structure corrections
  
  \[ U(r) = 4\pi Z \alpha \delta^3(r) \left( \frac{dG_P^E}{dq^2} \bigg|_{q^2=0} \right) = \frac{4\pi Z \alpha}{6} \delta^3(r)(r_E^p)^2 \]

- The change in the energy \( (m_r = m_\ell m_p / (m_\ell + m_p) \approx m_\ell) \)
  
  \[ \Delta E_{rE} = \int d^3r \psi(r)^\dagger U(r) \psi(r) = \frac{2\pi Z \alpha}{3} (r_E^p)^2 |\psi(0)|^2 \]
  
  \[ = \frac{2(Z\alpha)^4}{3n^3} m_r^3 (r_E^p)^2 \delta_\ell 0 \]

- Charge radius effects \( \propto m_r^3 \)
Charge radius from atomic physics

- Proton structure corrections

\[ U(r) = 4\pi Z\alpha \delta^3(r) \left( \frac{dG^p_E}{dq^2} \bigg|_{q^2=0} \right) = \frac{4\pi Z\alpha}{6} \delta^3(r)(r^p_E)^2 \]

- The change in the energy \( (m_r = m_\ell m_p/(m_\ell + m_p) \approx m_\ell) \)

\[ \Delta E_{r_E^p} = \int d^3r \psi(r)^\dagger U(r) \psi(r) = \frac{2\pi Z\alpha}{3} (r^p_E)^2 |\psi(0)|^2 \]

\[ = \frac{2(Z\alpha)^4}{3n^3} m_r^3 (r^p_E)^2 \delta_{\ell 0} \]

- Charge radius effects \( \propto m_r^3 \)

- Muonic hydrogen can give the best measurement of \( r^p_E \)!
CREMA Collaboration measured for the first time $2S_{1/2}^F = 1 - 2P_{3/2}^F = 2$ transition in Muonic Hydrogen
[Pohl et al. Nature 466, 213 (2010)]
Lamb shift in muonic hydrogen [Pohl et al. Nature 466, 213 (2010)]
\[ r^p_E = 0.84184(67) \text{ fm} \]
CODATA value [Mohr et al. RMP 80, 633 (2008)]
\[ r^p_E = 0.8768(69) \text{ fm} \]
extracted mainly from (electronic) hydrogen
Charge radius from atomic physics

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- $5\sigma$ discrepancy!
Lamb shift in muonic hydrogen \cite{Pohl2010}:
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\[ r_p^E = 0.8768(69) \, \text{fm} \]
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5\sigma discrepancy!

We can also extract it from electron-proton scattering data
The recent discrepancy

[1] Hill, GP PRD 82 113005 (2010) showed previous extractions are model dependent underestimated the error by a factor of 2 or more

Based on a model-independent approach using scattering data from proton, neutron and $\pi \pi$
\[ r_p^E = 0.871 \pm 0.009 \pm 0.002 \pm 0.002 \text{ fm} \]

Lamb shift in muonic hydrogen
\[ r_p^E = 0.84184(67) \text{ fm} \]

CODATA value (extracted mainly from electronic hydrogen)
\[ r_p^E = 0.8768(69) \text{ fm} \]
Lamb shift in muonic hydrogen

- CREMA collaboration measured for the *first time*
  - the Lamb shift in muonic hydrogen
  - Obviously very good experimentalists

\[ \Delta E = 206.2949 \pm 0.0032 \text{ meV} \]

Comparing to the theoretical expression
\[ \Delta E = 209.9779(49) - 5.2262(r_p E)^2 + 0.0347(r_p E)^3 \text{ meV} \]

They got \( r_p E = 0.84184(67) \text{ fm} \)

How reliable is the theoretical prediction?
Lamb shift in muonic hydrogen

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  Obviously very good experimentalists
  but unfortunately they need to relay on theorists to extract $r^p_E$...
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- They got
  $$r_E^p = 0.84184(67) \text{ fm}$$

- How reliable is the theoretical prediction?
The Theoretical Prediction

Is there a problem with the theoretical prediction?

\[ \Delta E = 209.9779(49) - 5.2262(r_E^p)^2 + 0.0347(r_E^p)^3 \text{ meV} \]

↑ mostly µ QED
↑ already discussed
↑ where does this term come from?

[Pachucki PRA 60, 3593 (1999), Borie PRA 71(3), 032508 (2005)]
Two-photon amplitude: “standard” calculation

- “standard” calculation
  - separate to proton and non-proton
  - non-proton ↔ DIS, polarizability
- For proton
  - Insert form factors into vertices

\[ M = \int_0^{\infty} dq^2 f(G_E, G_M) \]

- Using a “dipole form factor”

\[ G_E(q^2) \approx \frac{G_M(q^2)}{G_M(0)} \approx [1 - q^2/\Lambda^2]^{-2} \]

- \( M \) is a function of \( \Lambda \) \( \Rightarrow (r^D_E)^3 \) term
Two-photon amplitude: “standard” calculation

For proton
- Insert form factors into vertices
\[ \mathcal{M} = \int_0^\infty dq^2 f(G_E, G_M) \]
- Using a “dipole form factor”
\[ G_E(q^2) \approx G_M(q^2)/G_M(0) \approx [1 - q^2/\Lambda^2]^{-2} \]
- \( \mathcal{M} \) is a function of \( \Lambda \Rightarrow (r_E^p)^3 \) term

Using, for example, \( \Lambda^2 = 0.71 \text{ GeV}^2 \)
it contributes 0.018 meV to \( E(2p) - E(2s) \)
[K. Pachucki, PRA 53, 2092 (1996)]
Two-photon amplitude: “standard” calculation

- Is insertion of form factors in vertices valid?

- Even if it is, result looks funny
  two-photon amplitude ⇔ the charge radius
  only for one parameter model for $G_E$ and $G_M$
“Standard” Calculation: Summary

- Using
  \[ r_E^p = 0.871(10) \text{ fm} \quad \text{[Hill, GP PRD 82 113005 (2010)]} \]
  or
  \[ r_E^p = 0.8768(69) \text{ fm} \quad \text{[Mohr et al. RMP 80, 633 (2008)]} \]

- The measured interval in muonic hydrogen lies
  0.258(90)\,\text{meV} or 0.311(63)\,\text{meV} above theory.

- Using \( \Lambda^2 = 0.71 \text{GeV}^2 \),
  the proton contribution from the two-photon amplitude
  - 0.018\,\text{meV} to \( E(2p) - E(2s) \)
    \[ \text{[K. Pachucki, PRA 53, 2092 (1996)]} \]

- Is there a problem with the theoretical prediction?
NRQED

- Model Independent approach: use NRQED
- Up to $\mathcal{O}(1/m^3)$

[Caswell, Lepage PLB 167, 437 (1986); Kinoshita Nio PRD 53, 4909 (1996); Manohar PRD 56, 230 (1997)]

$$\mathcal{L}_e = \psi_e^\dagger \left\{ iD_t + \frac{D^2}{2m_e} + \frac{D^4}{8m_e^3} + c_F e \frac{\sigma \cdot B}{2m_e} + c_D e \frac{[\partial \cdot E]}{8m_e^2} \\
+ ic_S e \frac{\sigma \cdot (D \times E - E \times D)}{8m_e^2} + c_{W1} e \frac{\{D^2, \sigma \cdot B\}}{8m_e^3} \\
- c_{W2} e \frac{D^i \sigma \cdot B D^i}{4m_e^3} + c_{p'p} e \frac{\sigma \cdot DB \cdot D + D \cdot B \sigma \cdot D}{8m_e^3} \\
+ ic_{M} e \frac{\{D^i, [\partial \times B]^i\}}{8m_e^3} + c_{A1} e^2 \frac{B^2 - E^2}{8m_e^3} - c_{A2} e^2 \frac{E^2}{16m_e^3} \right\} \psi_e$$
NRQED

Need also

$$\mathcal{L}_{\text{contact}} = d_1 \frac{\psi_p^\dagger \sigma \psi_p \cdot \psi_e^\dagger \sigma \psi_e}{m_e m_p} + d_2 \frac{\psi_p^\dagger \psi_p \psi_e^\dagger \psi_e}{m_e m_p}$$

Matching

- Operators with one photon coupling:
  $$c_i \text{ given by } F_i^{(n)}(0)$$

- Operators with only two photon couplings:
  $$c_{A_i} \text{ given by forward and backward Compton scattering}$$

- $$d_i \text{ from two-photon amplitude}$$
Two-photon amplitude: matching

\[ \frac{1}{2} \sum_s i \int d^4x \ e^{iq \cdot x} \langle k, s | T \{ J_{e.m.}^\mu(x) J_{e.m.}^\nu(0) \} | k, s \rangle = \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) W_1 + \left( k^\mu - \frac{k \cdot q q^\mu}{q^2} \right) \left( k^\nu - \frac{k \cdot q q^\nu}{q^2} \right) W_2 \]

Matching

\[ \frac{4\pi m_r}{\lambda^3} - \frac{\pi m_r}{2 m_e m_p \lambda} - \frac{2\pi m_r}{m_p \lambda} \left[ F_2(0) + 4m_p^2 F_1'(0) \right] \]

\[ - \frac{2}{m_e m_p} \left[ \frac{2}{3} + \frac{1}{m_p^2 - m_e^2} \left( m_e^2 \log \frac{m_p}{\lambda} - m_p^2 \log \frac{m_e}{\lambda} \right) \right] + \frac{\delta d_2 (Z \alpha)^{-2}}{m_e m_p} \]

\[ = -\frac{m_e}{m_p} \int_{-1}^{1} dx \sqrt{1 - x^2} \int_{0}^{\infty} dQ \frac{Q^3}{(Q^2 + \lambda^2)^2(Q^2 + 4m_e^2 x^2)} \]

\[ \times \left[ (1 + 2x^2) W_1(2im_p Qx, Q^2) - (1 - x^2)m_p^2 W_2(2im_p Qx, Q^2) \right] \]
Relation between $\delta d_2$ and energy shift

$$\delta E(n, \ell) = -\delta_{\ell 0} \frac{m_r (Z \alpha)^3}{\pi n^3} \frac{\delta d_2}{m_e m_p}$$

In order to determine $\delta d_2$ need to know $W_i$

Im \[ \begin{array}{c} \includegraphics[width=0.2\textwidth]{figure1.png} \end{array} \] + \[ \begin{array}{c} \includegraphics[width=0.2\textwidth]{figure2.png} \end{array} \] $\sim$ Im $W_i$

can be extracted from on-shell quantities:
Proton form factor and Inelastic structure functions

In order to find $W_i$ need dispersion relations
but $W_1$ requires subtraction...
Dispersion relation

- Dispersion relations \((\nu = 2k \cdot q, \ Q^2 = -q^2)\)

\[
W_1(\nu, Q^2) = W_1(0, Q^2) + \frac{\nu^2}{\pi} \int_{\nu_{\text{cut}}(Q^2)^2}^{\infty} d\nu'^2 \frac{\Im W_1(\nu', Q^2)}{\nu'^2(\nu'^2 - \nu^2)}
\]

\[
W_2(\nu, Q^2) = \frac{1}{\pi} \int_{\nu_{\text{cut}}(Q^2)^2}^{\infty} d\nu'^2 \frac{\Im W_2(\nu', Q^2)}{\nu'^2 - \nu^2}
\]

- Decompose

\[
W_1(\nu, Q^2) = W_1(0, Q^2) + W_{1p,1}^{1}(\nu, Q^2) + W_{1c,1}^{1}(\nu, Q^2),
\]

\[
W_2(\nu, Q^2) = W_{2p,0}^{1}(\nu, Q^2) + W_{2c,0}^{1}(\nu, Q^2)
\]

- \(W_i^p\) from form factors
- \(W_i^c\) from DIS

- What about \(W_1(0, Q^2)\)?
Can calculate in two limits:

- $Q^2 \ll m_p^2$
  The photon sees the proton “almost“ like an elementary particle
  Use NRQED to calculate $W_1(0, Q^2)$ upto $\mathcal{O}(Q^2)$ (including)
  \[
  W_1(0, Q^2) = 2(c_F^2 - 1) + 2 \frac{Q^2}{4m_p^2} \left( c_{A_1} + c_F^2 - 2c_F c_{W_1} + 2c_M \right)
  \]

- $Q^2 \gg m_p^2$
  The photon sees the quarks inside the proton
  Use OPE to find $W_1(0, Q^2) \sim 1/Q^2$ for large $Q^2$

In between you will have to model!
Current calculation **pretends** that there is no model dependence
How big is the model dependence?
Bound states energies

- Convenient to talk about:
  - proton $W_i^p$, Continuum $W_i^c$, $W_1(0, Q^2)$

1) Proton $W_i^p$: using dipole form factor

\[ E(2p) - E(2s) = -0.016 \text{ meV} \]

2) Continuum [Carlson, Vanderhaeghen arXiv:1101.5965]

\[ E(2p) - E(2s) = 0.0127(5) \text{ meV} \]

3) What about $W_1(0, Q^2)$?

“Sticking In Form Factors” (SIFF) model

\[ W_1^{SIFF}(0, Q^2) = 2F_2(2F_1 + F_2) \quad F_i \equiv F_i(Q^2) \]
“Sticking In Form Factors” (SIFF) model

\[ W_{1}^{\text{SIFF}}(0, Q^2) = 2F_2(2F_1 + F_2) \quad F_i \equiv F_i(Q^2) \]

Notice that for large \( Q^2 \), \( W_{1}^{\text{SIFF}}(0, Q^2) \propto 1/Q^8 \)

In contradiction to OPE

There is no local Lagrangian that has a Feynman rule

\[ \gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}}{2m} F_2(q^2) q_\nu \]

Numerically using the dipole form factor

\[ \Delta E^{\text{SIFF}} = 0.034 \text{ meV} \]
Model Dependence

- How big is the model dependence?

\[ 0.018 \text{ meV} = -0.016 \text{ meV} + 0.034 \text{ meV} \]

\[ \uparrow \quad \uparrow \]

Model independent \quad Model dependent

- The model dependent piece is the dominant one!

- Experimental discrepancy \( \sim 0.3 \text{ meV} \)

- Can we find a model that explains (or reduces) the discrepancy?
New Physics?

- It is possible that the discrepancy is due to New Physics...

- New particle that couples to nucleons and $\mu$ (but not $e$ or $\tau$)
  
  Barger, Chiang, Keung, Marfatia [arXiv:1011.3519]

  **Assuming** same coupling to $\gamma$, $\eta$, $\pi$ rules this out

- New MeV particle that couples to protons ($g_p$) and muons ($g_\mu$)
  
  Tucker-Smith, Yavin [arXiv:1011.4922]

  Can explain $r_E^p$ and muon $g - 2$ but $g_p \approx g_n$ is problematic

- New $U(1)$ that couples only to right-handed muons
  
  Batell, McKeen, Pospelov [arXiv:1103.0721]
Conclusions

- Recent discrepancy in the extraction the proton charge radius between muonic and regular hydrogen
Conclusions

- Recent discrepancy in the extraction the proton charge radius between muonic and regular hydrogen.

- From model independent extraction of the charge radius from $e - p$ scattering data using the $z$ expansion.

$$r_E^P = 0.871 \pm 0.009 \pm 0.002 \pm 0.002 \text{ fm}$$ adding $\pi\pi$ data.

- Previous extractions have underestimated the error.

- Results are compatible with CODATA value of $r_E^P = 0.8768(69)$ fm.
Conclusions

- Analyzed Proton structure effects in hydrogenic bound states
  Using NRQED

- Isolated model-dependent assumptions in previous analyses:
  \( W_1(0, Q^2) \) was calculated by “Sticking In Form Factors” model

- Model independent calculation of \( W_1(0, Q^2) \):
  low \( Q^2 \) via NRQED, high \( Q^2 \) via OPE
  In between one has to model

- Possibility for a significant new effects in the two-photon amplitude

- NRQED predicts a universal shift for spin-independent
  energy splittings in muonic hydrogen.
Future Directions

- Applying $z$ expansion to the magnetic and axial-vector form-factors
- Analyze spin dependent effects
- Application to deuterium
- Resolution of the discrepancy?
Charge radius from Classic Lamb shift

- For electronic hydrogen: measured value Lunden and Pipkin '86

\[ E_{2s} - E_{2p_{1/2}} = 1.057845(9) \text{ GHz} = 0.00437490(4) \text{ meV} \]

compared to

\[ \Delta E_{r_p^E} = 0.0000008 (r_p^E)^2 \text{ meV/fm}^2 \]

Proton radius effects at a level of $10^{-4}$

Experimental uncertainty at a level of $10^{-5}$
Charge radius from Classic Lamb shift

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Proton radius effects at a level of $10^{-4}$
Experimental uncertainty at a level of $10^{-5}$

- For muonic hydrogen VP from electron loops dominant effect

\[ E_{2s} - E_{2p_{1/2}} \approx -205 \text{ meV} \]

compared to

\[ \Delta E_{r_p^E} = 5.2 (r_p^E)^2 \frac{\text{meV}}{\text{fm}^2} \]

Proton radius effects at a level of 2.5%
Experimental uncertainty at a level of $2 \times 10^{-5}$
Charge radius from Classic Lamb shift

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Experimental uncertainty at a level of $2 \times 10^{-5}$

- Muonic hydrogen can give the best measurement of $r_E^p$!
Two-photon amplitude: “standard” calculation

Is insertion of form factors in vertices valid?

Even if it is, result looks funny
two-photon amplitude ⇔ the charge radius
only for one parameter model for $G_E$ and $G_M$

Improvement?
Treat the two-photon amplitude as a new parameter

“Zemach” approximation: $m_l, \langle q \rangle \ll m_p$

[ Friar Annals Phys. 122, 151 (1979),
Eides et al. Theory of Light Hydrogenic Bound states, Springer ]

$$\Delta E = 209.9779(49) - 5.2262(r_E^P)^2 + 0.00913 \langle r^3 \rangle_{(2)} \text{ meV}$$
Third Zemach Moment

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\]

- The formula for the Lamb shift has two unknowns
  \( \Rightarrow \) use the CODATA value of \( r_E^P \) and solve for \( \langle r^3 \rangle_{(2)} \)
- The result \([\text{De Rújula PLB 693, 555 (2010)}]\]

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[\langle r^3 \rangle_{(2)}]^{1/3} = 3.32 \pm 0.21 \text{ fm } \text{muonic hydrogen}
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- Looks fine until we compare it to \( e - p \) scattering data
  \[ \langle r^3 \rangle_{(2)}^{1/3} = 1.39 \pm 0.02 \text{ fm} \quad \text{scattering data} \]

  [Sick, Friar PRA 72, 040502(R) (2005)]

Much more than \( 5\sigma \)… there is still a discrepancy
“Zemach” approximation: $m_l, \langle q \rangle \ll m_p$

but for $\Lambda^2 = 0.71 \text{GeV}^2$

$\Lambda \approx 0.84 \text{ GeV}$ is not small compared to $m_p$

Even worse

- Proton pole term

$$E(2p) - E(2s) = -0.016 \text{ meV}$$

- Using the Zemach approximation for proton pole term

$$E(2p) - E(2s) = +0.021 \text{ meV}$$

$\Rightarrow$ Thought to be an approximation only because $W_1^{\text{SIFF}}(0, Q^2)$!