Gravitational Path Integrals

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Abstract

This set of lecture notes is based on the first 3 weeks of a 10-week lecture course (PHYS 231C) given by Prof. Don Marolf at UCSB in the spring of 2021. It is intended as a pedagogical introduction to path integrals in quantum mechanics and gravity, aspects of classical gravity, and Euclidean techniques for gravitational entropy.

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0 Introduction and Conventions

Introduction to be written last.

Pre-requisites: classical mechanics (Lagrangian and Hamiltonian mechanics, and some ideas from Hamilton-Jacobi theory), quantum mechanics (path integrals to be reviewed), quantum statistical mechanics (in terms of density matrices), some QFT, and definitely GR (though many topics will be reviewed).

Conventions: natural units, but we keep $G$ and sometimes use $\hbar$ as a heuristic. The natural numbers start at zero, and the natural logarithm is $\ln$. Fourier transforms are unitary, i.e. they have factors of $\frac{1}{\sqrt{2\pi}}$. The spacetime dimension is $d$, so the spatial dimension is $d-1$. Bulk spacetime indices are early Latin, $abcd$, and boundary spacetime indices are middle Latin, $ijkl$. We occasionally use Greek indices $\alpha\beta\gamma\delta$ and $\mu\nu\rho\sigma$ for some calculations.
1 Path Integrals

1.1 The Canonical Path Integral (L1)

Our story begins with single-particle quantum mechanics in one dimension. The Hilbert space is \( \mathcal{H} = L^2(\mathbb{R}) \), and the system is governed by a self-adjoint Hamiltonian operator \( \hat{H} \). We work in the Heisenberg picture, where observables \( \hat{O} \) evolve in time according to

\[
\frac{d}{dt} \hat{O}(t) = i[\hat{H}, \hat{O}].
\]  

(1.1)

Now consider a one-parameter family of self-adjoint operators \( \hat{q}(t) \) and \( \hat{p}(t) \), the position and momentum, which have canonical equal-time commutator \( [\hat{q}(t), \hat{p}(t)] = i \). The Hamiltonian is allowed to depend on both \( \hat{q} \) and \( \hat{p} \), as well as explicitly on time: \( \hat{H} = \hat{H}(\hat{q}(t), \hat{p}(t), t) \). Note that for \( t < t' \), the commutator \( [\hat{q}(t), \hat{p}(t')] \) can only be obtained from the canonical one after solving the equation of motion: in other words, the commutator depends on \( \hat{H} \).

The path integral. The operator \( \hat{q}(t) \) has a complete set of eigenstates \( \{ |q, t \rangle \} \). These states are assumed to be orthonormal, in the sense that \( \langle q', t | q, t \rangle = \delta(q-q') \) at equal times. This does not tell us what the overlap \( K(q_i, t_i; q_f, t_f) \equiv \langle q_f, t_f | q_i, t_i \rangle \) is for \( t_i < t_f \), but surely this quantity, the propagator, depends on how \( \hat{H} \) evolves the system between times \( t_i \) and \( t_f \).

To construct the path integral that computes the propagator, we will proceed in four steps: (1) formally solve (1.1) for \( \hat{q}(t) \), and thereby relate the \( \hat{q} \)-eigenstates at times \( t_i \) and \( t_f \); (2) slice up the interval \([t_i, t_f]\) into a large number \( N \) of infinitesimal time steps, and resolve the identity at each step; (3) resolve a tricky ordering ambiguity in the Hamiltonian; and (4) take the continuum limit to obtain the path integral.

Step 1. The formal solution to the equation of motion (1.1) is given by conjugating the operator \( \hat{O} \) by the unitary time evolution operator \( U(t, t') \). This operator is constructed from the Hamiltonian by means of the Dyson series, which generalizes the familiar operator \( U(t) = e^{-i\hat{H}t} \) to the case of an arbitrary time-dependent Hamiltonian. For \( \hat{O} = \hat{q} \), we have

\[
\hat{q}(t_f) = U(t_i, t_f) \hat{q}(t_i) U(t_i, t_f)^{-1} = U(t_i, t_f) \hat{q}(t_i) U(t_f, t_i) = \mathcal{P} \exp \left[ i \int_{t_i}^{t_f} dt \hat{H}(t) \right] \hat{q}(t_i) \mathcal{P} \exp \left[ i \int_{t_f}^{t_i} dt \hat{H}(t) \right].
\]

(1.2)

Here \( \mathcal{P} \) is the path ordering symbol. Notice that inverting \( U \) is tantamount to reversing the path, so as to evolve backwards in time. This solution allows us to show that the eigenstates \( |q, t_i \rangle \) are related to \( |q, t_f \rangle \) by time evolution. The phase of the state simply rotates:

\[
|q, t_f \rangle = U(t_i, t_f) |q, t_i \rangle = \mathcal{P} \exp \left[ i \int_{t_i}^{t_f} dt \hat{H}(t) \right] |q, t_i \rangle.
\]

(1.3)

This is straightforward to check by applying \( \hat{q}(t_f) \) to the proposed form of \( |q, t_f \rangle \):

\[
\hat{q}(t_f) |q, t_f \rangle = U(t_i, t_f) \hat{q}(t_i) U(t_i, t_f)^{-1} U(t_i, t_f) |q, t_i \rangle = q U(t_i, t_f) |q, t_i \rangle = q |q, t_f \rangle. \quad \square
\]

(1.4)
1.1 The Canonical Path Integral (L1)

Step 2. We embark on the journey of computing the propagator:

\[ K(q_i, t_i; q_f, t_f) \equiv \langle q_f, t_f | q_i, t_i \rangle = \langle q_f, t_f | \mathcal{P} \exp \left[ i \int_{t_f}^{t_i} dt \, \hat{H}(t) \right] | q_i, t_i \rangle. \] (1.5)

Here the path has been reversed because we are time-evolving the dual vector \( \langle q_f, t_f \rangle \). The path-ordered exponential is defined as the following product:

\[ K(q_i, t_i; q_f, t_f) = \lim_{\Delta t \to 0} \left( \langle q_f, t_i | e^{-i\hat{H}(t)\Delta t} \ldots e^{-i\hat{H}(t')\Delta t} | q_i, t_i \rangle \right). \] (1.6)

The operator \( U(t', t) \) appearing here is a string of infinitesimal time evolution operators proceeding from \( t' \) to \( t \) in steps of size \( \Delta t \). Thus the rightmost \( \hat{H} \) is evaluated at \( t' \), the one before is evaluated at \( t' - \Delta t \), and so on until the leftmost \( \hat{H} \) is evaluated at \( t \). The minus signs above are responsible for undoing the backwards evolution of each step.

Next, we denote \( q_N \equiv q_f \) and \( q_0 \equiv q_i \), and insert \( N - 1 \) position-space resolutions of the identity between each exponential factor, where \( N = \frac{|t' - t|}{\Delta t} \) is the number of time steps:

\[
K(q_i, t_i; q_f, t_f) = \lim_{N \to \infty} \int dq_1 \ldots dq_{N-1} \langle q_N, t \rangle e^{-i\hat{H}(t)\Delta t} | q_N-1, t \rangle \langle q_N-1, t \rangle \ldots \\
\cdots | q_1, t \rangle \langle q_1, t | e^{-i\hat{H}(t')\Delta t} | q_0, t \rangle.
\] (1.7)

If we study just one of the factors above, we might try to work to first order in \( \Delta t \):

\[ K_1 \equiv \langle q_1, t | e^{-i\hat{H}(t')\Delta t} | q_0, t \rangle = \langle q_1, t | \left( 1 - i\hat{H}(t')\Delta t + O(\Delta t^2) \right) | q_0, t \rangle. \] (1.8)

But position-space matrix elements of \( \hat{H} \) are difficult to calculate in general: it is much easier to find matrix elements of \( \hat{H} \) sandwiched between position and momentum states. This motivates the insertion of yet another identity, this time in momentum space:

\[ K_1 = \int dp_1 \langle q_1, t | p_1, t \rangle \langle p_1, t | \left( 1 - i\hat{H}(t')\Delta t + O(\Delta t^2) \right) | q_0, t \rangle. \] (1.9)

At this point, we will assume that \( \hat{H} \) does not explicitly depend on time: \( \hat{H}(t) = \hat{H}(t') = \hat{H} \). (In fact, this restriction can be loosened with some more work: see [Don].)

Step 3. The problem with defining \( \hat{H} = \hat{H}(\hat{q}(t), \hat{p}(t)) \) as a “function” of the \( \hat{q} \) and \( \hat{p} \) operators is that there are ordering ambiguities due to noncommutativity. For instance, \( \hat{q}\hat{p} = \hat{p}\hat{q} + i \neq \hat{p}\hat{q} \). We will choose to define \( \hat{H} \) so that all \( \hat{p}'s \) are moved to the left and all \( \hat{q}'s \) are moved to the right. This convention is called Weyl ordering, and we define the Weyl symbol \( H_W(q, p) \) to be the following possibly complex-valued c-number function:

\[ H_W(q, p) = \frac{\langle p, t | \hat{H}| q, t \rangle}{\langle p, t | q, t \rangle}. \] (1.10)

We will actually take \( H_W \) to mean the real part of of the above, since the imaginary part is of order \( \hbar \) and vanishes in the semiclassical limit. See [Don] for more detail.
The numerator of $H_W$ is exactly what appears in the matrix element $K_1$ considered above:

$$K_1 = \int dp_1 \langle q_1, t | p_1, t \rangle \langle p_1, t | q_0, t \rangle \left( 1 - iH_W(q_1, p_1)\Delta t + O(\Delta t^2) \right). \quad (1.11)$$

Now, we can evaluate the integrand. The inner product wave functions $\langle q_1, t | p_1, t \rangle$ and $\langle p_1, t | q_0, t \rangle$ are plane waves, and (to first order) the $1 - iH_W\Delta t$ term is also an exponential:

$$K_1 = \frac{1}{2\pi} \int dp_1 e^{ip_1q_1} e^{-ip_1q_0} e^{-iH_W\Delta t} = \frac{1}{2\pi} \int dp_1 \exp \left( i[p(q_1 - q_0) - H_W(q_1, p_1)\Delta t] \right) =$$

$$= \frac{1}{2\pi} \int dp_1 \exp \left( i[p_1\dot{q}_1 - H_W(q_1, p_1)]\Delta t \right). \quad (1.12)$$

The factor of $\frac{1}{2\pi}$ comes from the $\frac{1}{\sqrt{2\pi}}$ normalization on each of the two plane waves. We have multiplied and divided by $\Delta t$ and introduced $\dot{q}_1 \equiv \frac{q_1 - q_0}{\Delta t}$. We recognize the term in brackets as the contribution to the (canonical) action on a small time interval $[t_0, t_1]$ of length $\Delta t$.

**Step 4.** We can now go back to the full propagator (1.7) and take the continuum limit $\Delta t \to 0$, or equivalently $N \to \infty$:

$$K(q_i, t_i; q_f, t_f) = \lim_{N \to \infty} \int dq_1 \cdots dq_{N-1} K_1 \cdots K_N =$$

$$= \lim_{N \to \infty} \left( \frac{1}{2\pi} \right)^N \int dq_1 \cdots dq_{N-1} dp_1 \cdots dp_N \exp \left( i \sum_{k=1}^N [p_k\dot{q}_k - H_W(q_k, p_k)]\Delta t \right) \equiv$$

$$\equiv \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq Dp \exp \left( i \int_{t_i}^{t_f} dt \left[ p\dot{q} - H_W(q, p) \right] \right). \quad (1.13)$$

The infinite-dimensional measure $Dq Dp$ on the set of phase space trajectories is a formal expression and is not rigorously defined. We introduce the **Lorentzian action** $S_L$,

$$K(q_i, t_i; q_f, t_f) = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq Dp e^{iS_L[q,p]}, \quad S_L[q, p] = \int_{t_i}^{t_f} dt \left( p\dot{q} - H_W \right). \quad (1.14)$$

### 1.2 Comments and Extensions (L1–L2)

**The measure.** In many cases, $S_L$ is quadratic in the momenta. In this case the path integral over $p$ is Gaussian and can be performed exactly. But this is not always true: more generally, if one actually manages to perform the path integral over momenta exactly, one will be left with a “measure factor” $\mu[q]$. To be precise, we write

$$K(q_i, t_i; q_f, t_f) = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq \mu[q] e^{iS_L[q,\dot{q}]} \quad (1.15)$$

One often says that properly accounting for the measure factor restores unitarity. However, the measure can safely be ignored (i.e. treated as $\mu = 1$) in the semiclassical approximation because its nontrivial dependence on $q$ is always of order $\hbar$. So we will sometimes ignore it.
Boundary conditions. Observe that the action $S_L$ produced by our derivation has precisely the correct form to yield a well-posed variational problem in which the initial and final positions are fixed. Fixing $q_i$ and $q_f$ also means that the measure $Dp$ includes one more momentum integral than $Dq$. (In this sense, the momenta live “between” positions.) We could instead have derived a path integral expression for $\langle p_f, t_f | p_i, t_i \rangle$. Such a path integral would have one more integral over $q$ than over $p$, and its action would be modified by a boundary term. This boundary term is responsible for well-posing a variational problem in which the initial momenta are fixed. We can also interpret the boundary term as part of a generating function that enacts a canonical transformation that rotates the $q$’s into the $p$’s.

General states. The discussion above admits a nice generalization. The boundary terms above can be regarded as coming from the wave functions of arbitrary states $|\psi\rangle$ and $|\psi'\rangle$. The path integral expression for their overlap is written

$$
\langle \psi' | \psi \rangle = \int dq_i dq_f \langle \psi' | q_f, t_f \rangle \langle q_f, t_f | q_i, t_i \rangle \langle q_i, t_i | \psi \rangle = \int dq_i dq_f \psi'(q_f, t_f) \psi(q_i, t_i) \langle q_f, t_f | q_i, t_i \rangle =
$$

$$
= \int Dq Dp \psi'(q_f) \psi(q_i) e^{iS_L} = \int Dq Dp \exp \left[ i(S_L - \ln \psi'(q_f) - \ln \psi(q_i)) \right].
$$

(1.16)

Here we absorbed the two position integrals $dq$ and $dq'$ into the path integral measure $Dq$. This unfortunate ambiguity in the notation is often left for readers to resolve. In the last expression, the $\ln \psi$ terms are not integrated $dt$, but are rather evaluated at the endpoints $q_i$ and $q_f$. They are boundary terms in $S_L$ and, as above, are responsible for canonical transformations. More poetically, if $|\psi\rangle$ is somewhere “between” $|q\rangle$ and $|p\rangle$, then then $\ln \psi$ terms in $S_L$ provide a suitable “phase rotation” of the variational problem to the $|\psi\rangle$ basis.

Correlators. Propagators are not the only quantities that path integrals can compute. They can also handle matrix elements of operators, or correlators. Fix instants in time $t_i < t_1 < t_2 < \cdots < t_n < t_f$ and consider operators $O(t_1), \ldots, O(t_n)$. Then by time-slicing and inserting complete sets of states between each operator, we arrive at

$$
\langle q_f, t_f | O(t_n) \cdots O(t_1) | q_i, t_i \rangle = \int Dq Dp O_W(t_n) \cdots O_W(t_1) e^{iS_L},
$$

(1.17)

where $O_W$ is the Weyl symbol of $O$. We often describe this situation by saying that we path-integrate from $t_i$ to $t_f$ and insert $O$ at $t_1, \ldots, t_n$. Note that the ordering prescription above only gives the correct expression for the correlator if the $O$ insertions are time-ordered.

Ordering and timefolds. More generally, we wish to consider out-of-time-ordered correlators (OTOCs), where the restriction $t_1 < \cdots < t_n$ is lifted. The key idea is to modify our time slicing. For example, suppose we want to compute $\langle q_f, t_f | O(t_3) O(t_2) O(t_1) | q_i, t_i \rangle$, where $t_i < t_3 < t_2 < t_1 < t_f$. To do so, we should start at $t_i$, evolve all the way up to $t_1$, insert $O(t_1)$, then evolve backwards to $t_2$, insert $O(t_2)$, evolve backwards again to $t_3$, insert $O(t_3)$, and finally evolve forward to finish at $t_f$. We draw this schematically in Fig. [1] below.
Writing down an explicit formula to describe this calculation would be extremely tedious; in practice, (1.17) is written down along with the picture above. The picture is equivalent to the following instructions: “compute a path integral of the form (1.14) on the interval \([t_i, t_f]\), evolving from one time step to the next using \(e^{-i\hat{H}\Delta t}\); then insert \(O(t_1)\); then compute another path integral on \([t_2, t_1]\), evolving from one time step to the next using \(e^{+i\hat{H}\Delta t}\); then insert \(O(t_2)\); etc.” This procedure is often called timefolding—the terminology is a nod to orbifolds. The main idea is just that path integrals compute path-ordered, rather than time-ordered, correlators, and we are free to specify the path by evolving forwards or backwards using \(e^{\pm i\hat{H}\Delta t}\), in order to hit all of the operator insertions in the right sequence.

Loops and traces. The object \(\langle q_f, t | O(t_1) | q_i, t \rangle\) is called an in-in or closed time path integral, because the initial and final times are equal, while (say) \(t_1 > t\). It is computed by the path integral depicted in Fig. 2 (left). If we also set \(q_i = q_f = q_0\) and integrate over all \(q_0\) as shown in Fig. 2 (right), the resulting path integral computes the trace of \(O(t_1)\):

\[
\int dq_0 \langle q_0, t | O(t_1) | q_0, t \rangle = Tr[O(t_1)] = \oint Dq Dp O_W(t_1) e^{iS_L},
\]

where the loop around the integral indicates periodic boundary conditions \(q(t_i) = q_0 = q(t_f)\). Notice that the trace depends only on \(t_1\), and is independent of the time \(t\) at which we started! More generally, it is a general property of path integrals that as long as we do not hit any operator insertions, the path integral is invariant under arbitrary deformations of the time-slicing contour. This is “obvious” if you don’t think too hard: going forward evolves by \(e^{-i\hat{H}\Delta t}\), going backward evolves by \(e^{+i\hat{H}\Delta t}\), and of course the two cancel each other out.

Preparing states. Occasionally, we wish to calculate overlaps \(\langle \psi' | \psi \rangle\) and correlators \(\langle \psi' | O | \psi \rangle\) in states that are prepared from some known vacuum or reference state \(|0\rangle\). That is, there are (possibly complicated) “guiding” Hamiltonians \(\hat{H}_g\) and \(\hat{H}_g'\) for which

\[
|\psi\rangle = \mathcal{P} \exp \left[ i \int_{-\infty}^{t_1} dt \hat{H}_g(t) \right] |0\rangle, \quad |\psi'\rangle = \mathcal{P} \exp \left[ i \int_{-\infty}^{t_f} dt \hat{H}_g'(t) \right] |0\rangle.
\]
Then to calculate \( \langle \psi' | \psi \rangle = \langle 0 | U(t_f, \infty) U(-\infty, t_i) | 0 \rangle \), we write down a path integral for \( \langle 0 | 0 \rangle \) on the entire real axis \( t \in (-\infty, \infty) \), being careful to evolve using \( \hat{H}_g \) on \( (-\infty, t_i] \) and using \( \hat{H}_g' \) on \( [t_f, \infty) \). This situation is depicted in Fig. 3 below. In words, we often say that the path integral “prepares” \( |\psi\rangle \) and \( |\psi'\rangle \), or that these states are assembled from sources.

![Figure 3: Preparing and evolving states using a path integral.](image)

### 1.3 Euclidean Path Integrals (L2)

**N.B.** From here on out we abbreviate “path integral” to PI.

**Euclidean time.** Suppose that we want to study the object \( \langle q_f, t_0 | e^{-\beta \hat{H}} | q_i, t_0 \rangle \), where \( \beta \) is a positive real number and we assume for simplicity that \( \hat{H} \) is time-independent. We can regard this as a special case of the in-in PI \( \langle q_f, t_0 | \mathcal{O} | q_i, t_0 \rangle \) for \( \mathcal{O} = e^{-\beta \hat{H}} \). Alternatively, if we make the somewhat ad-hoc identification \( \beta = i \Delta t \), the object above looks like an ordinary PI evolving in an imaginary time direction. Students of statistical mechanics will recognize \( e^{-\beta \hat{H}} \) as (proportional to) the thermal density matrix, which leads to the grandiose mantra that “statistical mechanics is quantum mechanics in imaginary time.”

In any case, we use \( \Delta t = -i \beta \) to write down a PI for this matrix element:

\[
\langle q_f, t_0 | e^{-\beta \hat{H}} | q_i, t_0 \rangle = \int_{q(t_0) = q_i}^{q(t_0) = q_f} \mathcal{D} q \mathcal{D} p \exp \left( i \int_{t_0}^{t_f} dt \left[ p \dot{q} - H_W \right] \right). \tag{1.20}
\]

Next, we define the imaginary time \( \tau = i(t - t_0) \), so that the change in \( \tau \) is \( \Delta \tau = i \Delta t = \beta \). (The transformation \( t \rightarrow i t \) is called a Wick rotation.) This allows us to rewrite the action by manipulating some factors of \( i \). We have \( d \tau = i dt \), and so \( \dot{q} = \frac{dq}{d \tau} = i \frac{dq}{dt} \). We then pull out another minus sign to define the Euclidean action \( S_E \):

\[
i S_L = i \int_{t_0}^{t_f} dt \left( p \frac{dq}{d t} - H \right) = - \int_{0}^{\beta} d \tau \left( -i p \frac{dq}{d \tau} + H_W \right) \equiv - S_E. \tag{1.21}
\]

Now using \( \dot{q} \) to refer to \( \frac{dq}{d \tau} \) for shorthand, the PI becomes

\[
\langle q_f, t_0 | e^{-\beta \hat{H}} | q_i, t_0 \rangle = \int_{q(0) = q_i}^{q(0) = q_f} \mathcal{D} q \mathcal{D} p \exp \left( - \int_{0}^{\beta} d \tau \left[ -i p \dot{q} + H_W \right] \right) = \int_{q(0) = q_i}^{q(0) = q_f} \mathcal{D} q \mathcal{D} p e^{-S_E}. \tag{1.22}
\]

**A simple example.** Consider the Hamiltonian for a single particle moving in one dimension under the influence of a potential \( V \). The Hamiltonian and its Weyl symbol are

\[
\hat{H} = \frac{p^2}{2m} + V(q) \quad \Rightarrow \quad H_W = \frac{p^2}{2m} + V(q). \tag{1.23}
\]
Our PI takes the form
\[
\langle q_f, t_0 | e^{-\beta \hat{H}} | q_i, t_0 \rangle = \int_{q(0)=q_i}^{q(0)=q_f} Dq \, Dp \, \exp \left( - \int_0^\beta d\tau \left[ -ip \dot{q} + \frac{p^2}{2m} + V(q) \right] \right). \tag{1.24}
\]

The \( Dp \) integral in (1.24) can be evaluated exactly because \( S_E \) is quadratic in \( p \). The \( p \) integral evaluates to a \( q \)-independent number, so the result is a covariant (Lagrangian), rather than a canonical (Hamiltonian), PI over positions, of the form (1.15) with \( \mu[q] = \mu \) a (formally infinite) constant. The covariant action \( S_E[q, \dot{q}] \) is obtained from the canonical one appearing in (1.24) by setting \( p \) to its equation of motion. This equation of motion is
\[
\frac{\delta S_E}{\delta p} = 0 = -i\dot{q} + \frac{\partial H_W}{\partial p} = -i\dot{q} + \frac{p}{m} \implies p = im\dot{q}. \tag{1.25}
\]
The Hamiltonian \( H_W = -\frac{mq^2}{2} + V(q) \) looks like it has the wrong sign on its kinetic term, a distinctive characteristic of Euclidean time. Meanwhile, the covariant Euclidean action is
\[
S_E[q, \dot{q}] = \int_0^\beta d\tau \left( m\dot{q}^2 - \frac{mq^2}{2} + V(q) \right) = \int_0^\beta d\tau \left( \frac{mq^2}{2} + V(q) \right). \tag{1.26}
\]
The integrand above looks distinctively energy-like. This leads to the continuation of our vague and grandiose mantra: “dynamics in real time is equivalent to statics in imaginary time.” The important point here is that since the “energy functional” \( S_E[q, \dot{q}] \) is positive-definite, the quantity \( e^{-S_E} \) has a convergent path integral.

**Thermal correlators.** Let us now take \( q_i = q_f = q_0 \) in the discussion above. As in (1.18), an extra integral over \( q_0 \) yields a path integral expression for the trace of \( e^{-\beta \hat{H}} \), an object known as the *partition function* in statistical mechanics:
\[
Z(\beta) = \text{Tr} \left[ e^{-\beta \hat{H}} \right] = \oint Dq \, Dp \, e^{-S_E}. \tag{1.27}
\]
Pictures are helpful here. Before, we drew path integrals by indicating horizontal real-time evolution arrows; now, as shown in Fig. 4, the arrows are vertical and denote evolution in imaginary time. One way to interpret the partition function is that the endpoints of the interval \([0, \beta]\) are identified, turning the interval into the *thermal circle* of circumference \( \beta \). It is denoted \( S^1(\beta) \), and its compactness is one of the many reasons Euclidean path integrals are nicer than their Lorentzian counterparts.

![Figure 4: Thermal correlators. From left to right: the \((q_i, q_f)\) matrix element of \( O \); the thermal expectation value of \( O \); the partition function; a three-point trace in complex time.](image-url)
Notice that if we take $\beta \to \infty$, the integrand rapidly decays to zero. The slowest decay is provided by the smallest value taken on by $S_\mathrm{E}$; this corresponds to the smallest eigenvalue of $\hat{H}$. Thus the limit $\beta \to \infty$ corresponds to a projection of the thermal density matrix $e^{-\beta \hat{H}}$ onto the ground state of the system, and the low-temperature properties of the system correspond to its ground-state dynamics. This happens irrespective of initial conditions, which is why we say that in equilibrium, any system settles into its ground state, and that the early-time fluctuations caused by the system’s initial conditions are washed out.

**Schwinger-Keldysh.** The Euclidean PI and its attendant pictures strongly suggest that we should think of $t$ as taking on complex values. This move allows us to draw arbitrary evolution contours in $\mathbb{C}$. So long as $\hat{H}$ is independent of time, it does not matter which contour we use to do PIs, so long as it passes through the right operator insertions in the right order. This partial analog of Cauchy’s theorem from complex analysis proves extremely useful, and allows for the calculation of thermal correlation functions of the form

$\text{Tr} \left[ e^{-\beta \hat{H}} O(t_1) \cdots O(t_n) \right], \quad t_1, \ldots, t_n \in \mathbb{C}$

by drawing a timefolded contour through $\mathbb{C}$ that passes through $t_1, \ldots, t_n$ in that order. These path integrals are called *Schwinger-Keldysh* PIs. But one must be careful: evolution with $e^{-\beta \hat{H}}$ “always” makes sense because $S_\mathrm{E}$ is usually positive-definite, but by the same token evolution with $e^{+\beta \hat{H}}$ “never” makes sense. One usually deals with this issue by ignoring it, or rather by trying hard never to decrease the imaginary part of $t$ in constructing PIs.

### 1.4 Gravitational Path Integrals (L3)

**Apologia.** Having studied path integrals in some generality, we seek to apply them to quantum gravity. To put things optimistically, no one knows how to do this. Being physicists, we will do so anyway. We will assume, for the most part, that our intuition from quantum mechanics carries over directly to quantum gravity, and that the resulting formal expressions are well-enough defined for us to do calculations with them. In this section, we will pause to take stock of the necessary generalizations and to “bemoan” the technical issues we will promptly leave behind afterwards. Thus this section consists primarily of vague platitudes, open ends, ill-posed questions, and the beginnings of several complicated stories.

**The quantum gravity PI.** We have considered the PI for a single particle moving in one dimension. But our formalism works equally well for any number of quantum particles moving on an arbitrary manifold. The next step is to generalize to the case of quantum field theory (QFT). This is usually done by considering a many-body system, often on a lattice, and taking the continuum limit of zero lattice spacing and infinitely many particles. Alternatively, one demotes position $q$ from the status of an operator to the status of a label, and then defines fields as operator-valued functions of all of the spacetime parameters. In this sense, quantum mechanics is a field theory whose “fields,” the operators $\hat{q}(t)$, live on a $(0 + 1)$-dimensional spacetime. After passing to field theory, one must grapple with the issue of gauge symmetry. All the while, one must ensure that the theory has a sensible Hamiltonian formulation; already at the level of gauge field theories, this is complicated.
One is then finally ready for quantum gravity. Crudely speaking, the “fields” comprise the spacetime metric $g_{\mu\nu}$, and one integrates $e^{iS}$ over all metrics to compute a gravitational PI. Whereas previously QFT was an account of fields fluctuating on a fixed spacetime background, in quantum gravity it is spacetime itself that is dynamical.

**The structure of gravity.** The idea of integrating $e^{iS}$ over all metrics begs two questions: (1) what, exactly, is $S$? and (2) if the metric $g$ is the “position” variable here, then what is the momentum? We will discuss (1) in detail soon; the terse answer is that $S$ is the Einstein-Hilbert action of general relativity (GR) with the Gibbons-Hawking-York boundary term. We will not discuss (2) in great detail. There are several approaches to a Hamiltonian formulation of GR, among them the ADM and first-order (Cartan) formalisms. Each framework comes with its own subtleties and difficulties, many of them related to gauge invariance.

**Gauge issues I.** As mentioned above, classical general relativity (GR) is a gauge theory. This means that there are transformations—changes one can make to the action—that leave the action invariant. Any two field configurations related by a gauge transformation are regarded as physically identical, and the action evaluated on two gauge-related configurations is the same. If $\mathcal{G}$ is the group of gauge transformations of the theory, then the set of all field configurations related to each other by the action of elements of $\mathcal{G}$ is called a *gauge orbit*. Starting from any field configuration $\phi$, applying a transformation $g \in \mathcal{G}$ will produce a distinct configuration $g\phi$ in the orbit of $\phi$, which is moreover unique and distinct from $\phi$. Thus each $\mathcal{G}$-orbit is isomorphic to $\mathcal{G}$ itself, and the space $\mathcal{A}$ of all field configurations is foliated (i.e. ruled) by the gauge orbits. The physical degrees of freedom in a gauge theory are thus labeled by the gauge orbits, so the phase space of the theory is $\mathcal{P} = \mathcal{A}/\mathcal{G}$.

One might imagine coordinatizing $\mathcal{P}$ by gauge invariant variables $q_{\text{inv}}$ and $p_{\text{inv}}$; one might further imagine using the fact that the action is constant along the gauge orbits to construct a PI by integrating the quantity $e^{iS}$ over $\mathcal{P}$. In practice, we do this by *gauge fixing*: we choose a representative configuration on each gauge orbit and coordinatize the resulting “slice” through the gauge orbits, as shown in Fig. 5 below.

![Figure 5: The space $\mathcal{A}$ of field configurations. Each gauge orbit (dotted) is isomorphic to the gauge group $\mathcal{G}$, and the gauge slices (solid) are isomorphic to the phase space $\mathcal{P} = \mathcal{A}/\mathcal{G}$.](image-url)
**1.4 Gravitational Path Integrals (L3)**

**Gauge issues II.** In all but the simplest gauge theories, it is hard to find a global set of such coordinates. It is usually easier to integrate over all of $\mathcal{A}$. The fact that $S$ is constant along gauge orbits manifests itself as a prefactor of the (infinite) volume of the gauge group:

$$\int Dq Dp \, e^{iS} = \int Dq_{\text{inv}} Dp_{\text{inv}} D(\text{orbits}) \, e^{iS} = \text{vol}(\mathcal{G}) \int Dq_{\text{inv}} Dp_{\text{inv}} e^{iS}. \quad (1.29)$$

On one hand, what we really want to study is the naïve path integral over metrics and momenta, divided by this volume; for the most part, this is what we will do. Ignoring gauge symmetry like this sometimes has funny consequences: if two $q \neq q'$ lie on the same gauge orbit, then “obviously” $\langle q', t \mid q, t \rangle \neq 0$, even though the notation would suggest otherwise.

On the other hand, extremely sophisticated technology exists for handling the true gauge invariant path integral. All of these techniques stem from the Faddeev-Popov approach to gauge fixing and from BRST symmetry. Among them, the BFV, BV, and antifield formalisms are the most powerful; they are also extremely abstract. It should be noted that the infamous *Gribov ambiguity* often prevents one from being able to find global coordinates on $\mathcal{P}$, i.e. from fixing a gauge slice that intersects each orbit exactly once. This problem is especially manifest in the case of gravity. To make matters worse, the gauge group in GR is the infinite-dimensional diffeomorphism group: every coordinate transformation is gauged.

For the most part, we will be able to evade all of these issues by working semiclassically. This amounts to the declaration that the gauge-invariant PI is exactly equal to $e^{iS_0}$, where $S_0$ is the value of the gravitational action evaluated on a stationary point, i.e. on a classical solution. By formally evaluating the PI, we simply assume that gauge fixing and all its discontents have come and gone; what remains after the dust clears is just classical GR.

**Indefiniteness.** The Euclidean gravitational action, which we will discuss soon, is *not* positive-definite! In fact, it is unbounded above and below. We offer two terse proofs. The first proof: the Einstein-Hilbert Lagrangian consists essentially of the scalar curvature $R$ of a spacetime, and by choosing arbitrarily curved spacetimes like spheres or hyperbolic spaces, one can make $R$ and hence the gravitational action arbitrarily positive or negative. The second proof: recall the Friedman equation from cosmology,

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + \frac{8\pi G \rho}{3}. \quad (1.30)$$

Suppose the matter content of the universe is a spatially homogeneous, massless scalar field $\phi(t)$. Then the energy density $\rho$ includes a kinetic term $\dot{\phi}^2$. Then the Friedman equation reads $\dot{\phi}^2 - (\dot{a}/a)^2 + \cdots = 0$, which tells us that the scale factor contributes a negative “gravitational” kinetic energy and makes the total energy unbounded.

**The Gauss law.** We saw in the previous section that the positive-definiteness of “sensible” Euclidean actions helps the operator $e^{-\beta H}$ yield a convergent PI. So the situation here looks bad—really bad. But energy in GR is extremely subtle: the gravitational Hamiltonian is in fact positive-definite, but is only so on shell, i.e. when the equations of motion hold. Actually, one only needs to satisfy the *constraint equations*, which impose gauge invariance and are analogous to the Gauss law in electromagnetism.
The cosmological example above actually shows that the total energy (and thus the Hamiltonian) of the system is always constrained to be exactly zero. In fact, this is also what happens in electromagnetism, where Gauss’s law forces the total charge in a closed universe to vanish. So the Euclidean action is unbounded, but the on-shell Hamiltonian is positive-definite. This suggests that a truly gauge invariant formulation of GR should produce a manifestly positive Euclidean action, and that somehow the process of dropping the constraints and losing gauge invariance is responsible for the unboundedness of $S_E$. (Another way to say this is that we should constrain, then quantize, rather than quantize, then constrain.) Two approaches to this problem, which is called the conformal factor problem in the literature, are typically taken: (1) work in the semiclassical approximation, which is already on shell and thus avoids these issues; or (2) Wick-rotate the part of the metric responsible for the scale factor to improve the convergence of the path integral.

**The Euclidean action.** The (Lorentzian) Einstein-Hilbert action for a spacetime $(M,g)$ is the integral of its scalar curvature over the spacetime manifold:

$$ S_L[g] = \frac{1}{16\pi G} \int_M d^dx \sqrt{-g} R, \quad (1.31) $$

One immediate problem with defining a corresponding Euclidean action is that arbitrary pseudo-Riemannian manifolds generically lack a global time coordinate to Wick-rotate, so it is unclear what the prescription $t \rightarrow \tau = it$ means. We will return to this issue soon; for the moment let us assume that it all makes sense. The Wick-rotated spacetime is then a Riemannian manifold whose metric and scalar curvature are also denoted $g$ and $R$.

The Euclidean gravitational action looks almost identical to the Lorentzian action, except for an unexpected minus sign:

$$ S_E[g] = -\frac{1}{16\pi G} \int_M d^dx \sqrt{g} R. \quad (1.32) $$

This sign is really just a consequence of carrying out the Wick rotation carefully and consistently imposing $i S_L = -S_E$. It can be seen most clearly in the spacetime $M = S^3 \times \mathbb{R}$, where $\mathbb{R}$ is the time axis. Here $R_L = R_E = R^{(3)}$ is the constant curvature of the 3-sphere. Then upon integrating over a finite time interval $[t_i, t_f]$, $t = -i\tau \Leftrightarrow dt = -i d\tau$ gives

$$ i S_L = \frac{i}{16\pi G} \int_M d^4x \sqrt{-g} R_L = \frac{i \text{vol}(S^3)}{16\pi G} R^{(3)} \int_{t_i}^{t_f} dt = (i)(-i) \frac{\text{vol}(S^3)}{16\pi G} R^{(3)} \int_{\tau_i}^{\tau_f} d\tau = + \frac{1}{16\pi G} \int d^4x \sqrt{g} R_E = -S_E \implies S_E = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R. \quad (1.33) $$

**Topology change.** In classical GR, the best-behaved spacetimes are called globally hyperbolic. These manifolds have the topology $M = \Sigma \times \mathbb{R}$, where $\mathbb{R}$ is the time axis, and the Cauchy surface $\Sigma$ retains the same topology at all times. It is this class of spacetimes for which the initial value problem in GR is well posed, for which the Hamiltonian formalism is well defined, and on which QFT in curved spacetime is most naturally formulated. In particular, globally hyperbolic spacetimes have a globally defined time coordinate, and this makes it more clear what a Wick rotation means.
A big question arises: should the gravitational PI integrate over spacetimes with arbitrary topology? In Lorentzian signature, where topology change is classically forbidden in the sense described above, the answer seems to be “no; restrict to globally hyperbolic manifolds.” In Euclidean signature, however, a plethora of topologies exist which have finite action and satisfy the Einstein equations. The question of which spacetimes to include in the gravitational PI is still open; in the remainder of these notes, we will take the view that the gravitational PI does sum over all topologies. The consequences will be dramatic.

**Pants.** One justification for this choice comes from the fact that Hamilton-Jacobi theory (in the semiclassical approximation) can be used to show that Euclidean solutions in gravity are related to tunneling processes in the Lorentzian theory. This seems to indicate that there are topology-changing processes even in the Lorentzian theory. A cute illustration of this idea is based on the (Euclidean) pair of pants spacetime, shown in Fig. 6.

![Figure 6: Pants.](image)

We imagine the time arrow as running upwards and consider spatial slices of the surface. There are initially two disjoint universes; as they coalesce into a larger universe, the spatial geometry is momentarily singular at the crossover point. Even though the Euclidean spacetime is perfectly smooth, it does not admit a smooth Lorentzian metric. Rather than discarding such metrics, however, we will learn later on how to deal with singularities in the gravitational PI. We sometimes think of the lower half of the pair of pants as the configuration space of 2-universe states, and the upper half as the configuration space of 1-universe states. They are joined at their boundary by a singular configuration, which nevertheless should have finite action, which suggests that it can be tunneled through.

In retrospect, this should not be too surprising. The PI measure is generically supported on singular configurations, paths (or geometries) which are not smooth, nor even $C^1$. 
2 Gravity in a Box

2.1 The Gravitational Action (L4)

Motivation. After developing the canonical path integral in quantum mechanics in both Lorentzian and Euclidean signature, we discussed its generalization to the case of gravity. For the most part, our PI calculations will be semiclassical, which means that we must evaluate the gravitational action on solutions to Einstein’s equations. It therefore behooves us to understand the classical action of GR in more detail. We will do so here in Lorentzian signature, and in the next section in Euclidean signature.

The variational principle. In most GR courses, one is taught that the Einstein field equations of general relativity follow from the Einstein-Hilbert action \( (1.31) \). This is indeed true, but the derivation involves throwing away certain boundary terms:

\[
S_{EH}[g] = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} R \implies \delta S_{EH} = \int_M d^4x \sqrt{-g} (\text{EOM})_{ab} \delta g^{ab} + \text{(boundary terms)}.
\]  

These terms do not affect local physics in the interior of \( M \), so often the Einstein-Hilbert action is good enough. But we are interested in the actual numerical value of the action, so we must tread carefully to make sure we are using the correct action to begin with.

We say that an action has a good variational principle or yields a well-posed variational problem when its variation gives only the equations of motion, with no extra boundary terms. Such boundary terms may be interpreted as “unwanted” equations of motion on the boundary; moreover, the discussion of semiclassical PIs above applies only to actions with a good variational principle. The Einstein-Hilbert action evidently does not satisfy this requirement, so we must add to it an additional boundary term that cancels the one above.

Gravity in a box. For pedagogical simplicity, we will take the spacetime \((M, g)\) to be the “box” shown in Fig. [left]. We assume that \( M \) is bounded to the future and past by initial and final surfaces, and to the “sides” by a timelike (i.e. spatial) boundary manifold \( \partial M \). Depending on which boundary conditions we choose to impose and which variational problem we wish to solve, the aforementioned boundary term in the action will live on either the spatial or past/future boundaries. For the standard Cauchy problem in GR, the boundary term lives only on the spatial boundary. This is a declaration by fiat rather than a claim to be proven; here are two pieces of intuition that help explain why we do it.

First of all, we will soon see that this boundary prescription is equivalent to the claim that the Hamiltonian in the canonical formulation of GR is a boundary term; hence it generates time translations on \( \partial M \). In other words, the Hamiltonian defines the gravitational energy of \( M \) by generating a canonical transformation that moves the future and past boundaries while preserving the spatial boundary. All of this befits a variational problem where the action is specified on the fixed spatial boundary, but not on the “movable” temporal boundaries. Yet another equivalent way to say this is that modifying the action at the spatial boundaries modifies the Hamiltonian and therefore changes the system under consideration.
States. Second of all, recall the discussion of section 1.2 where it was shown that the boundary terms appearing in the action are not universal, but are rather motivated by the particular quantum states whose overlap we want to compute using a PI,

$$\langle \psi' | \psi \rangle = \int Dq DP \exp \left[ i (S_L - \ln \psi'(t_f)^* - \ln \psi(t_i)) \right], \quad (2.2)$$

where—crucially—$|\psi\rangle$ and $|\psi'\rangle$ are specified on the past and future boundaries of $M$. Therefore changing the action on the past and future boundaries is tantamount to modifying the state of the system, while changing the action on the spatial boundary is equivalent to modifying the system itself. We aim to fix the physical system and examine the evolution of states, so we should fix the action on the spatial boundaries.

As a side comment, (2.2) shows that the true variational principle that defines the instantiation of the classical theory we care about is not $\delta S_L = 0$, but rather

$$\delta(S_L - \ln \psi'(t_f)^* - \ln \psi(t_i)) = 0 \iff \delta S_L = \ln \left[ \psi'(t_f)^* \psi(t_i) \right]. \quad (2.3)$$

In other words, an “action flux” at the past and future boundaries is allowed, so long as we understand it to represent a modification of the initial and final states of the system.

A look ahead. In the following subsection, we will make the discussion above more precise. We will begin by varying the Einstein-Hilbert action explicitly to better understand (2.1). Next, we discuss the choice of boundary conditions on $\partial M$ that well-defines the variational problem we aim to solve. We then define and meditate on the extrinsic curvature as the key tool for constructing the requisite boundary term. And finally, after a few more comments, we arrive at the celebrated Gibbons-Hawking-York boundary term in the gravitational action. For the impatient reader who wishes to skip §2.2, the end result is

$$S = S_{EH} + S_{GHY} = \frac{1}{16\pi G} \int_M d^d x \sqrt{-g} R + \frac{1}{8\pi G} \int_{\partial M} d^{d-1} x \sqrt{-h} K, \quad (2.4)$$

where $h$ is the induced metric on $\partial M$ and $K$ is the extrinsic curvature of $\partial M \subset M$.

Figure 7: Left: gravity in a box. Right: the extrinsic curvature.
2.2 Boundary Terms and Conditions (L4)

Variation of the action. A calculation, found (e.g.) in Appendix E of [Wald], shows that
\[
\delta S_{EH} = \frac{1}{16\pi G} \int_M d^d x \sqrt{-g} \left( R_{ab} - \frac{1}{2} R g_{ab} \right) \delta g^{ab} + \frac{1}{16\pi G} \int_{\partial M} d^{d-1} x \sqrt{-h} n^a_{\partial M} v_a. \tag{2.5}
\]
Here \( h \) is the (Lorentzian) boundary metric on \( \partial M \) induced from the bulk metric \( g \), \( n_{\partial M} \) is
the unit normal to \( \partial M \), and \( v \) is the variation given by
\[
v^a = (-g^{ab} g^{cd} + g^{bd} g^{ac}) \nabla_b (\delta g_{cd}). \tag{2.6}
\]
At first sight, \( v \) looks like a complete mess. But on closer inspection, dimensional analysis
almost completely fixes its form. The Ricci scalar contains two derivatives of the metric, so
the variation and an integration by parts removes one:
\[
R \sim \partial \partial g \implies \delta R \sim \nabla (\delta g) + \delta (\nabla g).
\]
The resulting object has three indices and must be contracted with the one-index unit normal,
so it needs to be an object that carries four indices and has no metric derivatives. The only
two options for such a term are exactly what appear in parentheses above. The only thing
not fixed by this argument is the numerical prefactors at each term.

Dirichlet and Neumann. From the index contractions present in (2.5–2.6), we can see
that the first term of \( n^a_{\partial M} v_a \) involves only derivatives of the metric in the normal direction to
\( \partial M \), while the second term involves derivatives in both the normal and tangential directions.

Now, in order to specify the variational problem to be solved by the full gravitational
action, we need to choose the boundary conditions obeyed by the metric field on \( \partial M \). Two
common choices are Dirichlet conditions, whereby \( h \) is held fixed, and Neumann conditions,
whereby the normal derivatives of \( h \) are held fixed. We will focus on Dirichlet boundary
conditions: we set \( \delta h = 0 \), and as a consequence all derivatives of \( \delta g \) tangent to \( \partial M \) must
vanish. Therefore the boundary term we seek must have a variation which, subject to
\( \delta h = 0 \), produces exactly the right normal-derivative contributions to cancel (2.6). More explicitly,
the full gravitational action must take the form
\[
S_L = S_{EH} + S_\partial = \frac{1}{16\pi G} \int_M d^d x \sqrt{-g} R + \frac{1}{16\pi G} \int_{\partial M} d^{d-1} x \sqrt{-h} \mathcal{L}_\partial, \tag{2.7}
\]
where \( \mathcal{L}_\partial \) is a scalar on \( \partial M \) constructed only from the boundary metric \( h \) using at most one
normal derivative. What can such a term look like?

Extrinsic curvature. While objects familiar from GR like the Riemann tensor measure
the curvature of a manifold intrinsically, without reference to an embedding in some larger
ambient space, the extrinsic curvature tensor directly measures the curvature of a submanifold
as embedded within a larger space. The extrinsic curvature is a naïve notion: a cylinder
in 3D space, for example, has vanishing Riemann tensor, but is “obviously” not flat relative
to the ambient \( \mathbb{R}^3 \), and indeed has positive extrinsic curvature.

Given an embedded codimension-1 submanifold \((N,h)\) of \((M,g)\), its extrinsic curvature
\( K \) is the trace of half of the (outward-pointing) normal Lie derivative of the induced metric:
\[
K_{ij} = \frac{1}{2} \mathcal{L}_n h_{ij}, \quad K = K_{ij} h^{ij}. \tag{2.8}
\]
Let’s unpack this definition in the case \(N = \partial M\): see Fig. 7 (right) above. The unit normal \(n^a\) to \(\partial M\) is a vector field defined only on \(\partial M\), but we can extend it to a vector field \(\tilde{n}^a\) in the bulk of \(M\) in an arbitrary (but smooth) way. Although it is not yet obvious, the following is actually independent of \(\tilde{n}^a\). Viewing \(\tilde{n}^a\) as the infinitesimal generator of a diffeomorphism, we imagine moving \(\partial M\) along the flow lines of \(\tilde{n}^a\): each point on \(\partial M\) moves by the same proper distance in the normal direction. We can also use \(\tilde{n}^a\) to transport the boundary metric \(h\) to the new location of \(\partial M\). (This is called “pulling back the metric.”) The transported metric \(h'\) on the flowed boundary \(\partial M'\) can then be compared to the restriction of the bulk metric \(g\) to the surface \(\partial M'\). The difference between the two, evaluated infinitesimally close to \(\partial M\), is the Lie derivative \(\mathcal{L}_n h_{ij}\). In this way, \(n^a\) provides a measure of how “crinkly” \(\partial M\) is, and that crinkliness is what the extrinsic curvature picks up.

Example: the sphere. Let \(S^n(r) \subset \mathbb{R}^{n+1}\) be the \(n\)-sphere of radius \(r\). The unit normal vector points radially outwards, \(n = \partial_r\), and the metric induced on \(S^n\) is the usual round metric \(ds^2 = h_{ij} dx^i dx^j = r^2 d\Omega_n^2\). Under an infinitesimal variation in the radial direction, i.e. extending to \(\tilde{n} = \partial_r\) away from \(S^n(r)\), we find

\[
ds^2 = r^2 d\Omega_n^2 \implies \delta(ds^2) = 2r \, d\Omega_n^2 \delta r \implies \delta h_{ij} = 2r \delta r.
\]

The normal Lie derivative and the extrinsic curvature are therefore

\[
\mathcal{L}_n h_{ij} = \frac{\delta h_{ij}}{\delta r} = 2r \, d\Omega_n^2 = \frac{2}{r} h_{ij} \implies K_{ij} = \frac{1}{r} h_{ij} \implies K = \frac{1}{r} h_{ij} h^{ij} = \frac{n}{r}.
\]  

Some useful tools. Here we collect some formulæ that we will not discuss in detail, but which prove useful in practice and whose derivations can be found in textbooks like \(\text{[Wald]}\).

First, we define an object \(h_{ab}\), with bulk indices in \(M\) and closely related to the boundary metric \(h_{ij}\), by \(h_{ab} = g_{ab} - n_a n_b\). Formally, \(h_{ab}\) is the pullback of the boundary metric from \(\partial M\) to \(M\) along the natural projection \(p: M \to \partial M\). In particular, raising an index gives the matrix \(h^b_a\) of (the differential of) this projection. The main property of \(h_{ab}\) is that it annihilates the normal direction: \(n^a h_{ab} = n_b - (n^a n_b) n_b = 0\). Conversely, if \(v^a\) and \(w^b\) are tangent to \(\partial M\), then contraction with \(h_{ab}\) leaves them unchanged: \(v^a h_{ab} w^b = v^a w_a = v^a h_{ij} w^j\).

In other words, \(h_{ab}\) is something like a projection on to the directions tangent to \(\partial M\).

It turns out that \(h_{ab}\) appears in a very practical formula for computing the (pullback of the) extrinsic curvature: \(K_{ab} = h^c_a \nabla_c n_b\). This formula avoids the explicit use of Lie derivatives, and makes manifest the claim that \(K_{ab}\) depends only on the normal vector to \(\partial M\), and not on its extension into \(M\). As with \(h_{ab}\) above, \(K_{ab}\) has bulk indices and is related to \(K_{ij}\) in the same tangential-projection manner as \(h_{ab}\) is to \(h_{ij}\). It therefore satisfies \(K = K_{ab} g^{ab} = K_{ab} h^{ab} = K_{ij} h^{ij}\). Finally, we note that the extrinsic curvature tensor \(K_{ij}\), the Riemann tensor \(R_{abcd}\), and the boundary Riemann tensor \(R^M_{ijkl}\) are not independent! They are related by the Gauss-Codazzi equations (see \(\text{[Wald]}\)), but we will not need them here.

The Gibbons-Hawking-York term. The extrinsic curvature is essentially the boundary term we need. We define the Gibbons-Hawking-York term to be

\[
S_{\text{GHY}} = \frac{1}{8\pi G} \int_{\partial M} d^{d-1}x \sqrt{-h} K.
\]
It can then be shown, by a somewhat tedious calculation, that

$$\delta (S_{EH} + S_{GHY}) = \int_M d^d x \sqrt{-g} (EOM)_{ab} \delta g^{ab} - \frac{1}{16\pi G} \int_{\partial M} d^{d-1} x \sqrt{-h} \Pi^{ij} \delta h_{ij}, \quad (2.12)$$

where $\Pi^{ij} \equiv K^{ij} - Kh^{ij}$. If we take $S_{EH} + S_{GHY}$ to be the gravitational action, then the calculation above shows that its variation produces, aside from the equations of motion, a boundary term that vanishes upon imposing the Dirichlet boundary condition $\delta h_{ij} = 0$. The quantity $\Pi^{ij}$ is sometimes called the gravitational momentum, and on spacelike surfaces, it is the variable canonically conjugate to the metric (in the sense of Hamilton-Jacobi theory). As we shall see, it is also closely related to the Brown-York boundary stress tensor. So now we are done: by taking $S = S_{EH} + S_{GHY}$ and specifying $\delta h = 0$ on $\partial M$, we have set up a good variational principle for gravity in a box.

2.3 The Brown-York Stress Tensor (L4)

Background structures.

The Brown-York stress tensor.

Covariant conservation.

Conserved charges.
3 Euclidean Gravity

3.1 Euclidean Path Integrals (L5)

3.2 Energy and Entropy (L6)

3.3 The Hawking-Page Transition (L6)
A References

[Don’s document] Corrections to Week 1 lectures

[Wald] GR textbook