Covariant Phase Space
Classical Field Theory Done Right

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The physicist’s mind yearns for **field theory**: it is beautiful.

The physicist’s heart yearns for **phase space**: it is elegant.

The physicist’s soul yearns for **covariance**: it is sublime.
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**Theorem (my strongest conviction)**

*Both the Lagrangian (variational) and Hamiltonian (symplectic) formalisms work best when all three of these principles are unified.*
Executive Summary

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The **variational principle** and the **symplectic potential**:

\[ \delta \mathcal{L} = \mathcal{E} \delta \phi + \nabla_\mu \theta^\mu, \quad \theta^\mu = \pi^\mu \delta \phi. \]  (0.1)

The **symplectic form** and **Hamilton’s equations**:

\[ \omega^\mu = \delta \theta^\mu = \delta \pi^\mu \wedge \delta \phi, \quad \Omega = \int_\Sigma n_\mu \omega^\mu, \quad \iota_X \Omega = \delta \mathcal{H}. \]  (0.2)
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Applications: CPS is unreasonably beautiful and unifies classical physics. It reproduces the ADM mass and reveals BH entropy as a Noether charge. It has the capacity to understand the phase space of GR, whose degrees of freedom live “on the boundary.”
Outline

1. The Problem of Time
2. Particle Mechanics
   - The Variational Principle
   - Hamiltonian Mechanics
3. Particles and Fields
4. Covariant Phase Space
   - Theme and First Variation
   - Example: Free Scalar
   - Diffeomorphism Charges
5. Gravity at Last
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The Hole Argument

Suppose that we solve the initial value problem in GR for $g(x)$.

Perform a coordinate transformation $\psi: M \rightarrow M$, sending $x \mapsto y = \psi(x)$, which leaves the initial value surface $\Sigma$ fixed.
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Perform a coordinate transformation $\psi: M \rightarrow M$, sending $x \mapsto y = \psi(x)$, which leaves the initial value surface $\Sigma$ fixed.

By coordinate invariance, $g(y) = (\psi^*g)(x) \neq g(x)$ must also solve the same initial value problem. Thus $g$ is not determined uniquely!
The Hole Argument

**The problem:** GR seems to be indeterministic.

**The resolution (physics):** the solutions \((M, g)\) and \((M, \psi^* g)\) are gauge-equivalent by the active diffeomorphism \(\psi \in \text{Diff}(M)\).

**The resolution (math):** the spacetimes \((M, g)\) and \((\psi(M), \psi^* g)\) are isometric by \(\psi\), by virtue of pulling back the metric.
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**The moral:** one must be careful when speaking of time, since the concept is generally meaningless. The initial value problem is not a covariant notion, and can be approached only in special cases.

**Warning:** this applies equally to all spacetimes, not just the weird ones. (And by the way, AdS is not even globally hyperbolic.)
The Problem of Time

The Hamiltonian in GR is zero. Three ways to see this:

1. **Physics:** the evolution of $g$ is locally indistinguishable from a gauge transformation, which has vanishing Noether charge.
2. **Math:** $H = 0$ for any reparametrization-invariant system.
3. **Philosophy:** $g$ is dynamical; it *creates* spacetime. There is no prior geometry, so $g$ cannot evolve with a parameter it makes.
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3. **Philosophy**: $g$ is dynamical; it creates spacetime. There is no prior geometry, so $g$ cannot evolve with a parameter it makes.

We conclude that time in GR does not flow; it just *is*.

Meanwhile, unitarity in QM demands an absolute, rigid, external notion of time: $U = e^{-i\hat{H}t}$, and $|\psi(t)\rangle = U(t)|\psi_0\rangle$. As long as QM embraces an evolution parameter, it cannot be fully covariant.
Blindly quantizing yields the **Wheeler-DeWitt equation**, the Schrödinger equation for the quantum state of the universe:

\[ \hat{H} |\Psi\rangle = 0. \]  

(1.1)

The wave function of the universe has no universe in which to evolve. It lives in the Hilbert space of quantum metric fields.
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**QM issues:** The WDE has no classical limit (i.e. no \( \hbar \)). The state \( |\Psi\rangle \) is “frozen” and cries out for a background-independent QM.

**GR issues:** \( H \) is the wrong thing to consider! We need a covariant object that generates the phase space flow of the metric.

We turn to the classical phase space of field theory and of gravity. Is there any more noble goal than to geometrize geometry?
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Let $M \subset \mathbb{R}^n$ be the world. The **trajectory** of a particle is a curve $q: \mathbb{R}_t \longrightarrow M$, and local coordinates $q^i \in \mathbb{R}^n$ describe its position.
Trajectories and Velocities

Let $M \subset \mathbb{R}^n$ be the world. The **trajectory** of a particle is a curve $q: \mathbb{R}_t \rightarrow M$, and local coordinates $q^i \in \mathbb{R}^n$ describe its position.

Its velocity is a vector $(q, v)$ in the **tangent space** $T_q M$, which has a natural basis $\left\{ \frac{\partial}{\partial q^i} = \partial_i \right\}$. Thus $v = v^i \frac{\partial}{\partial q^i} = v^i \partial_i$. 
“Why are \( q \) and \( \dot{q} \) treated as independent?”

- The numbers \( v^i \) are coefficients needed to specify an arbitrary \( v \in T_q M \), and can be chosen independently of \( q^i \) (duh).
- But when \( v \) is actually tangent to the trajectory, \( v^i = \dot{q}^i (t) \).
- The symbol \( \dot{q}^i \) was the *name* historically given to \( v^i \). Nice job.
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The initial data $(q_0, \dot{q}_0)$ uniquely determine $q(t)$. But they also determine $\dot{q}(t)$. So $q(t)$ and $\dot{q}(t)$ effect each other’s dynamics.

Hence we are interested in the particle’s combined trajectory $(q, \dot{q}) : \mathbb{R}_t \rightarrow TM$ traced out through the tangent bundle $TM$.

**How does one determine this trajectory?**
The **Lagrangian** of a mechanical system is a function $L : TM \rightarrow \mathbb{R}$, and the **action** functional is its integral over $\mathbb{R}_t$:

$$S[q(t), \dot{q}(t)] = \int_{\mathbb{R}} dt \, L(q, \dot{q}).$$  \hspace{1cm} (2.1)

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By requiring that $\delta q = 0$ at infinity, the **Euler-Lagrange (EL) equations** follow. In local coordinates on $TM$, these are

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = 0.$$  

(2.2)

**E.g.** For a free particle on $(M, g)$, the Lagrangian is the metric, $L(x, \dot{x}) = \frac{1}{2} g(\dot{x}, \dot{x}) = g_{ij} \dot{x}^i \dot{x}^j$. The EOM is the geodesic equation.
Upgrading to Differential Forms

Everything in sight is now a **differential form**—an antisymmetric tensor—on the parameter space $\mathbb{R}_t$ (time, basis $dt$, “horizontal”) as well as on the target space $M$ (space, basis $\delta q^i$, “vertical”).

*E.g.* $L = \mathcal{L} dt$ is a 1-form on $\mathbb{R}_t$ and a 0-form on $M$. The action $S = \int_{\mathbb{R}} L$ is a scalar. The variation $\delta L$ is then a $(1, 1)$-form.
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**Our main tools:** \( d^2 = \delta^2 = 0 \) (“differential forms are fermions”) and **Stokes’s theorem**, \( \int_M d\omega = \int_{\partial M} \omega \) (duality of \( d \) and \( \partial \)).

**E.g.** when the EOM hold, the variation of \( L \) must either vanish or be a total time derivative with vanishing integral over \( \mathbb{R}_t \):

\[
\delta L = (\delta \mathcal{L}) dt = \left( \frac{d\sigma}{dt} \right) dt = d\sigma, \quad \int_{\mathbb{R}} d\sigma = \int_{\partial \mathbb{R}} \sigma = 0. \quad (2.3)
\]
Consider a single particle moving in one dimension. We vary $L$:

$$
\delta L = (\delta \mathcal{L}) dt = \left[ \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right] \delta q \ dt + \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right) dt \equiv \\
\equiv \mathcal{E} \delta q \ dt + d(p \delta q) \equiv d\sigma.
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The $(1, 0)$-form $E = \mathcal{E} dt$ is called the Euler-Lagrange form.

The $(0, 1)$-form $\theta \equiv p \delta q$ is called the symplectic potential.
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The $(1,0)$-form $E = \E dt$ is called the Euler-Lagrange form. The $(0,1)$-form $\theta \equiv p \delta q$ is called the symplectic potential.

At the level of the action, this calculation reads

$$\delta S = \int_{\mathbb{R}} \delta L = \int_{\mathbb{R}} E \delta q dt + \int_{\partial \mathbb{R}} p \delta q = 0. \quad (2.5)$$
Both the Euler-Lagrange equations and Noether’s theorem follow from \( \delta L = E \delta q + d(p \delta q) = d\sigma \) by setting different terms to zero.
Main Results of Lagrangian Mechanics

Both the Euler-Lagrange equations and Noether’s theorem follow from $\delta L = E \delta q + d(p \delta q) = d\sigma$ by setting different terms to zero.

1. If $\delta q$ vanishes on the boundary, i.e. $\delta q \big|_{\partial R} = 0$, then demanding $\delta S = 0$ gives us the equations of motion:

$$\delta S = \int_{\mathbb{R}} \mathcal{E} \delta q \, dt + \int_{\partial R} p \delta q \, = 0 \implies \mathcal{E} \equiv 0.$$

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Both the Euler-Lagrange equations and Noether’s theorem follow from \( \delta L = E \delta q + d(p \delta q) = d\sigma \) by setting different terms to zero.

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\]

2. If \( \delta q \) is an on-shell symmetry, i.e. \( \delta S = 0 \) when \( \mathcal{E} = 0 \),

\[
\delta L = \mathcal{E} \delta q \, dt + \frac{d}{dt} (p \delta q) \, dt = \frac{d\sigma}{dt} \, dt \implies \frac{d}{dt} (p \delta q - \sigma) = 0.
\]

We call \( p \delta q - \sigma \) the Noether current of the symmetry \( \delta q \).
Example: The Harmonic Oscillator

The configuration space is $\mathbb{R}_x$, and the tangent bundle is $\mathbb{R}^2_{(x,\dot{x})}$.

The Lagrangian is $L(x, \dot{x}) = \mathcal{L} \, dt = \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right) dt$, so

$$
\delta L = \left[ -m \omega^2 x - m \ddot{x} \right] \delta x \, dt + \frac{d}{dt} \left( m \dot{x} \delta x \right) \, dt.
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The equations of motion are

$$\mathcal{E} = -m \omega^2 x - m \ddot{x} = 0 \iff \ddot{x} = -\omega^2 x, \quad (2.8)$$

and the symplectic potential is $\theta = p \delta x = m \dot{x} \delta x$. 
A Noether current is obtained for each symmetry $\delta x$ of the action.

**E.g.** $\delta x = \dot{x} \implies \delta \dot{x} = \ddot{x}$ generates time translation. We have:

$$
\delta L = \left[ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right] dt = \left[ (-m\omega^2 x)(\dot{x}) + (m\dot{x})(\ddot{x}) \right] dt = \\
= m(\dot{x}\dddot{x} - \omega^2 x\dot{x}) dt = \frac{d}{dt} \left[ \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 \right] dt = dL. \quad (2.9)
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Thus $\delta L = dL$ and $\theta = m\dot{x} \delta x = m\dot{x}^2 = p \delta x$, so $\mathcal{H}$ is conserved:

$$\frac{d}{dt} (p \delta x - \mathcal{L}) = \frac{d}{dt} \left( \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega^2 x^2 \right) = \frac{d\mathcal{H}}{dt} = 0. \quad (2.10)$$

We have “discovered” the Legendre transformation of $\sigma = L$ via $\theta$. 
Four Treatments of Hamiltonian Mechanics

We will use a combination of the following methods:

1. Legendre transforms. Assemble Hamilton's equations by considering $\delta L$ and $\delta H$. Straightforward and accessible, but unenlightening and obscures the symplectic structure.
2. Conservation of energy. Consider a Hamiltonian vector field. Physically motivated, but not (immediately) symplectic.
3. Construct phase space. Use $\theta$ to build the symplectic form. Possibly illuminating, but is a long story and takes effort.
4. Mathematics. Make the answer a definition and prove that it works. Deep and precise, but unmotivated and too abstract.

The uncomfortable truth: it works because it works.
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**Hamilton’s Equations for Dummies**

**Big idea:** The phase space trajectory $\gamma(t) = (q(t), p(t))$ is an integral curve of the Hamiltonian vector field $X = (\dot{q}, \dot{p})$. 

\[
\begin{align*}
\nabla H \\
X = \dot{\gamma} \\
\gamma(t)
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Hamilton’s Equations for Dummies

**Big idea:** The phase space trajectory $\gamma(t) = (q(t), p(t))$ is an integral curve of the **Hamiltonian vector field** $X = (\dot{q}, \dot{p})$.

The Hamiltonian should be conserved. Thus $\gamma$ lies on a level surface of constant $H(q, p) = E$, and $X$ is orthogonal to $\nabla H$:

$$(\dot{q}, \dot{p}) = X \perp \nabla H = \left( \frac{\partial H}{\partial q}, \frac{\partial H}{\partial p} \right). \quad (2.11)$$
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Therefore the components of $X$ must be $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$.
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What is Phase Space?

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The action of \( p \) on \( v \in TM \) is

\[
p(v) = (p \delta q) \left( \dot{q} \frac{\partial}{\partial q} \right) = p \dot{q}.
\]

If \( v = \frac{d}{dt} q(t) \), this is \( \theta \) again!
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The canonical 1-form $\theta$ projects $X = (\dot{q}, \dot{p}) \in T\mathcal{M}$ to $\dot{q} \in TM$ and feeds the result to $p$.

Since $\theta$ is also $p \delta q$, it is the bridge between the Lagrangian and Hamiltonian viewpoints.
We are now in a position to use $\theta = p \delta q$ to relate $X$ to $H = \mathcal{H} \, dt$. The key insight is that $X \perp \nabla \mathcal{H} \sim \delta \mathcal{H}$; to make this precise, we seek an antisymmetric machine that raises and lowers indices.
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Hamiltonian Mechanics

**The Symplectic Form**

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The **symplectic form** is $\omega = \delta \theta = \delta p \wedge \delta q$. It is closed, $\delta \omega = 0$, and nondegenerate, i.e. $\iota_X \omega \equiv \omega(X, -)$ is a 1-form unique to $X$.

Now $\omega(X, -)$ “lowers the index” of $X$, sends $(\frac{\partial}{\partial q}, \frac{\partial}{\partial p}) \mapsto (\delta q, \delta p)$, and rotates its entries by $\frac{\pi}{2}$. (Un)surprisingly, the result is $-\delta \mathcal{H}$:
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$$X = (\dot{q}, \dot{p}) \implies \iota_X \omega = (-\dot{p}, \dot{q}) = -\delta \mathcal{H} = \left(-\frac{\partial \mathcal{H}}{\partial q}, -\frac{\partial \mathcal{H}}{\partial p}\right). \quad (2.12)$$

Thus **Hamilton’s equations** are expressed by $\iota_X \omega + \delta \mathcal{H} = 0$. 


Example: The Harmonic Oscillator

The configuration space is $\mathbb{R}_t$, and the phase space is $\mathbb{R}^2_{(p,x)}$.

The Hamiltonian is $H(p, x) = \left(\frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2\right) dt = \mathcal{H} dt$.

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Now we assemble Hamilton’s equations. We have

$$\delta \mathcal{H} = \frac{\partial \mathcal{H}}{\partial x} \delta x + \frac{\partial \mathcal{H}}{\partial p} \delta p = m\omega^2 x \delta x + \frac{p}{m} \delta p,$$

$$X = (\dot{x}, \dot{p}) = \dot{x} \frac{\partial}{\partial x} + \dot{p} \frac{\partial}{\partial p} \implies \iota_X \omega = -\dot{p} \delta x + \dot{x} \delta p. \quad (2.13)$$
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The configuration space is \( \mathbb{R}_t \), and the phase space is \( \mathbb{R}^2_{(p,x)} \).

The Hamiltonian is \( H(p, x) = \left( \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2 \right) dt = \mathcal{H} \ dt \).

The symplectic form is \( \theta = p \delta x \implies \omega = \delta \theta = \delta p \wedge \delta x \).

Now we assemble Hamilton’s equations. We have

\[
\delta \mathcal{H} = \frac{\partial \mathcal{H}}{\partial x} \delta x + \frac{\partial \mathcal{H}}{\partial p} \delta p = m \omega^2 x \delta x + \frac{p}{m} \delta p,
\]

\[
X = (\dot{x}, \dot{p}) = \dot{x} \frac{\partial}{\partial x} + \dot{p} \frac{\partial}{\partial p} \implies \iota_X \omega = -\dot{p} \delta x + \dot{x} \delta p. \quad (2.13)
\]

Therefore \( \iota_X \omega = -\dot{p} \delta x + \dot{x} \delta p \overset{!}{=} -m \omega^2 x \delta x - \frac{p}{m} \delta p = -\delta \mathcal{H} \),

and matching differentials yields \( \dot{x} = \frac{p}{m} \) and \( \dot{p} = -m \omega^2 x \). Nice!
Outline

1 The Problem of Time

2 Particle Mechanics
   • The Variational Principle
   • Hamiltonian Mechanics

3 Particles and Fields

4 Covariant Phase Space
   • Theme and First Variation
   • Example: Free Scalar
   • Diffeomorphism Charges

5 Gravity at Last
Summary So Far

Lagrangian mechanics:

- $L = \mathcal{L} \, dt$ lives on $TM$ and determines the path $(q(t), \dot{q}(t))$.
- The variational principle yields the EOM and Noether charges.
- The all-important symplectic potential $\theta = p \, \delta q \sim \delta S$ typically vanishes on $\partial \mathbb{R}$, but can be nonzero in the bulk.
- Its variation $\omega = \delta \theta = \delta p \wedge \delta q \sim \delta^2 S$ is the symplectic form.
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Hamiltonian mechanics:

- The Hamiltonian vector field $X$ generates phase space flow and determines the path $\gamma(t) = (q(t), p(t))$ through $\mathcal{H} = E$.
- We reimagine $\theta$ as the canonical 1-form on $T^*M$.
- The symplectic form encodes the structure of Hamilton’s equations, and converts between $X$ and $H$. 
Two Schools of Thought

A **particle** is a curve $q : \mathbb{R}_t \rightarrow M$ with position $q(t) = x \in M$. 
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**Fields are particle densities.** *Space and time are different.*

- In a fog of $N \rightarrow \infty$ particles $q^a(t)$, the index $a \rightarrow x$ ranges over $M$. The fog’s density is a scalar field $\phi(x) = \phi(x, t)$.
- The field has one degree of freedom at every $x \in M$, and the notion of particle positions evaporates: $M^M \rightarrow M \times \mathbb{R}$.
- This is radical, unwieldy, and infinite-dimensional. It looks hard to motivate non-scalar fields or make things covariant.
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**Fields are sigma models.** *Space and time are unified.*

- A **field** \( \phi: M \longrightarrow F \) maps spacetime points \( x \in M \) to field values \( \phi(x) \in F \), and has \( \dim F \) degrees of freedom.
- This generalizes the time parameter and the configuration space of particle mechanics to arbitrary manifolds.
First Attempt: De Donder–Weyl Theory

Lagrangian field theory is already covariant: \[ \frac{\partial L}{\partial \dot{\phi}} = \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} \phi)} \right). \]

It is tempting to “do the same thing” in Hamiltonian field theory.
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It is tempting to “do the same thing” in Hamiltonian field theory. The polymomenta \( \pi^\mu \) and De Donder–Weyl Hamiltonian are

\[
\pi^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}, \quad \mathcal{H} = \pi^\mu \partial_\mu \phi - \mathcal{L}. \tag{3.1}
\]

The De Donder–Weyl equations are the “obvious” ones:

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\partial_\mu \phi = \frac{\partial \mathcal{H}}{\partial \pi^\mu}, \quad \partial_\mu \pi^\mu = -\frac{\partial \mathcal{H}}{\partial \phi}. \tag{3.2}
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N.B. This \( \mathcal{H} \) is covariant, but does not generate time translations; meanwhile, the “textbook” \( \mathcal{H} \) cannot be covariant! Also, the DW theory has too many momenta. The CPS formalism soaks up the index in \( \pi^\mu \) by choosing a Cauchy surface \( \Sigma \) and considering \( \pi^\mu n_\mu \).
Too Many Bundles: Fields and their Jets

We generalize the tangent bundle, spanned by vectors $\dot{q}$, to a space spanned by *all* of field derivatives $\partial_\mu \phi$. This is the jet bundle $J^1 F$. We also want to consider both spacetime differentials $dx$ and field variations $\delta \phi$, so we define the field bundle $E \longrightarrow M$ with fiber $F$. 
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\textbf{Lagrangian FT} takes place on $J^1 E$, spanned by $(x^\mu, \phi^a, \partial_\mu \phi^a)$. \textbf{Hamiltonian FT} happens on $(J^1 E)^*$, spanned by $(x^\mu, \phi^a, \pi^\mu, a)$.

Directions in $M$ (base/spacetime/input/source) are \textbf{horizontal}. Directions in $F$ (fiber/field-space/output/target) are \textbf{vertical}.
We generalize the tangent bundle, spanned by vectors $\dot{q}$, to a space spanned by all of field derivatives $\partial_\mu \phi$. This is the jet bundle $J^1 F$. We also want to consider both spacetime differentials $dx$ and field variations $\delta \phi$, so we define the field bundle $E \rightarrow M$ with fiber $F$.

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Directions in $M$ (base spacetime/input/source) are horizontal. Directions in $F$ (fiber field-space/output/target) are vertical.

E.g. The real scalar field: $M = \mathbb{R}^{3,1}_{x^\mu}$, $F = \mathbb{R}_\phi$, and $E = \mathbb{R}^{3,1}_{x^\mu} \times \mathbb{R}_\phi$. Then $(J^1 E)^* = \mathbb{R}^{3,1}_{x^\mu} \times \mathbb{R}_\phi \times \mathbb{R}^{3,1}_{\pi^\mu}$. Everything is finite-dimensional!
How Classical Physics Should Be Done

On a spacetime $M^n$, $L = \mathcal{L} \varepsilon_M$ becomes an $(n, 0)$-form. The Euler-Lagrange equations are encoded in the interactions between the horizontal and vertical derivatives $\mathcal{d}$ and $\delta$. 
How Classical Physics Should Be Done

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The de Rham differential on $J^1E$ is $d = d + \delta$, and its $(d, \delta)$-bigraded de Rham complex is the variational bicomplex.

The fundamental calculation is then the equality of $(n, 1)$-forms

$$dL = \delta \mathcal{L} \varepsilon_M = \mathcal{E} \varepsilon_M \delta \phi - d\theta = d\sigma. \quad (3.3)$$
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The fundamental calculation is then the equality of $(n,1)$-forms

$$dL = \delta \mathcal{L} \epsilon_M = \mathcal{E} \epsilon_M \delta \phi - d\theta = d\sigma. \quad (3.3)$$

The **multisymplectic potential** and **multisymplectic form** are

$$\theta = (\pi^\mu \delta \phi) n_\mu \epsilon_{\partial M} \in \Omega^{(n-1,1)}, \quad \omega = \delta \theta \in \Omega^{(n-1,2)}. \quad (3.4)$$

**The dream**: do geometrical quantization to all of this!
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The Road Ahead

We turn to the construction of the **covariant phase space**:
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We turn to the construction of the *covariant phase space*:

1. Begin with the **kinematic configuration space** $\mathcal{C}$ and its **dynamical shell** $\tilde{\mathcal{P}}$, also called the pre-phase space.

2. Vary the action, **taking care of boundary conditions**, to obtain a pre-symplectic potential $\tilde{\theta}$ and pre-symplectic form $\tilde{\Omega} = \delta\tilde{\theta}$.

3. Quotient out $\tilde{\mathcal{P}}$ and $\tilde{\Omega}$ by **gauge symmetries**, which are zero modes of $\tilde{\Omega}$. This gives us the covariant phase space $(\mathcal{P}, \Omega)$.

4. Given a Hamiltonian vector field $X_\xi$, construct the corresponding **diffeomorphism charge** $H_\xi$. 
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Once all of this is done, we will proceed to apply it to gravity!
In moving from particles to fields, we put *volume form* on $M$:

$$
dt \rightarrow \varepsilon_M = \sqrt{-g} \, d^n x = \sqrt{-g} \, dx_{\mu_1} \wedge \cdots \wedge dx_{\mu_n}. \quad (4.1)$$
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If $n$ is the outward unit normal form/vector to $\partial M$, then the volume form on $\partial M$ is given by $\varepsilon_M = n \wedge \varepsilon_{\partial M} \iff \varepsilon_{\partial M} = \iota_n \varepsilon_M$. 
Volume Forms and Boundaries

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E.g. On the half-Minkowski space $\mathbb{R}^{3,1}_{x \leq 0}$, we have

$$\varepsilon_M = dt \wedge dx \wedge dy \wedge dz,$$

$$n^\mu = \partial^x = (0, 1, 0, 0),$$

$$\varepsilon_{\partial M} = n^\mu \, dx_\mu \, dx_\nu \, dx_\rho \, dx_\sigma = -dt \wedge dy \wedge dz. \tag{4.2}$$
Variation of the Lagrangian

We proceed as before: given \( L = \mathcal{L}(\phi, \partial_\mu \phi) \, \varepsilon_M \), we have

\[
\delta L = \left[ \frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \left( \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi)} \right) \right] \delta \phi \, \varepsilon_M + \nabla_\mu \left( \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi)} \delta \phi \right) \varepsilon_M = \\
= \mathcal{E} \, \delta \phi \, \varepsilon_M + \nabla_\mu (\pi^\mu \delta \phi) \varepsilon_M \overset{!}{=} (\nabla_\mu \sigma^\mu) \varepsilon_M. \tag{4.3}
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$$= \mathcal{E} \delta \phi \epsilon_M + \nabla_\mu (\pi^\mu \delta \phi) \epsilon_M = \left( \nabla_\mu \sigma^\mu \right) \epsilon_M. \quad (4.3)$$

We get a vector’s worth of symplectic potentials $\theta^\mu = \pi^\mu \delta \phi$ and variations $\sigma^\mu$. The “product rule” gives us their boundary values:

$$\left( \nabla_\mu \sigma^\mu \right) \epsilon_M = d\sigma,$$

$$\sigma \bigg|_{\partial M} = (n_\mu \sigma^\mu) \epsilon_{\partial M},$$

$$\left( \nabla_\mu \theta^\mu \right) \epsilon_M = d\theta,$$

$$\theta \bigg|_{\partial M} = (n_\mu \theta^\mu) \epsilon_{\partial M}. \quad (4.4)$$

As advertised, $\pi^\mu n_\mu$ is the “correct” momentum conjugate to $\phi$. 


At the level of the action, these divergences become surface terms:

\[ \delta S = \int_M \mathcal{E} \delta \phi \varepsilon_M + \int_M (\nabla_\mu \theta^\mu) \varepsilon_M = E + \int_M d\theta = E + \int_{\partial M} \theta = E + \int_{\partial M} (n_\mu \pi^\mu \delta \phi) \varepsilon_{\partial M}. \]  

(4.5)

If \( \delta \phi \) vanishes on \( \partial M \), then \( \delta S = 0 \) implies \( \mathcal{E} = 0 \) as usual.
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If \(\delta \phi\) vanishes on \(\partial M\), then \(\delta S = 0\) implies \(\mathcal{E} = 0\) as usual. And if \(\delta \phi\) is an on-shell symmetry, we get Noether’s theorem:

\[
\delta L = (\nabla_\mu \theta^\mu) \varepsilon_M = (\nabla_\mu \sigma^\mu) \varepsilon_M \implies \nabla_\mu (\theta^\mu - \sigma^\mu) = 0.
\]

(4.6)

Thus the **Noether current** \(j^\mu = \pi^\mu \delta \phi - \sigma^\mu\) is conserved.
The “full” symplectic potential $\theta$ and form $\omega$ are defined by contracting $\theta^\mu$ and $\omega^\mu = \delta \theta^\mu$ into $\varepsilon_M$:

$$\theta^\mu = \pi^\mu \delta \phi \implies \theta = \nu_{\theta^\mu} \varepsilon_M \implies (n_\mu \pi^\mu \delta \phi) \varepsilon_{\partial M},$$

$$\omega^\mu = \delta \pi^\mu \wedge \delta \phi \implies \omega = \nu_{\omega^\mu} \varepsilon_M \implies (n_\mu \delta \pi^\mu \wedge \delta \phi) \varepsilon_{\partial M}. \quad (4.7)$$
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$$
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To obtain a Hamilton equation $\iota_X \omega = -\delta \mathcal{H}$, one tries to write down $X \sim (\nabla \phi, \nabla \pi)$ generalizing $(\dot{q}, \dot{p})$. But this proves unwieldy.
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To obtain a Hamilton equation $\nu_X \omega = -\delta \mathcal{H}$, one tries to write down $X \sim (\nabla \phi, \nabla \pi)$ generalizing $(\dot{q}, \dot{p})$. But this proves unwieldy.

We restrict to globally hyperbolic $M$, choose a Cauchy surface $\Sigma$, call $\omega$ the *symplectic density*, and define the *symplectic form*

$$\Omega = \int_\Sigma \omega = \int_\Sigma (\hat{n}_\mu \omega^\mu) \varepsilon_\Sigma. \quad (\Omega_\Sigma = \Omega_{\Sigma'}) \quad (4.8)$$

where $\hat{n}$ is the (past-pointing) normal to $\Sigma$. This is still covariant!
The real, free scalar field on Minkowski spacetime $M = \mathbb{R}^{3,1}$ has phase space $(J^1 E)^* = \mathbb{R}^{3,1}_{x\mu} \times \mathbb{R}_\phi \times \mathbb{R}^{3,1}_{\pi\mu}$ and Lagrangian

$$L = \mathcal{L} \, d^4x = - \left[ \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} m^2 \phi^2 \right] d^4x.$$  \hspace{1cm} (4.9)
The real, free scalar field on Minkowski spacetime $M = \mathbb{R}^{3,1}$ has phase space $\mathcal{J}_1 E^\ast = \mathbb{R}_{x^\mu}^3 \times \mathbb{R}_\phi \times \mathbb{R}_{\pi^\mu}^3$ and Lagrangian

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The conjugate momenta are $\pi^\mu = \partial^\mu \phi$. We vary the Lagrangian to obtain the equations of motion and the symplectic data:

$$\delta L = \left[ (\frac{\partial_\mu \partial^\mu - m^2}{\mathcal{E}}) \phi \right] \delta \phi \, d^4 x + \delta \phi \left( \partial^\mu \phi \delta \phi \right) \, d^4 x. \quad (4.10)$$
The real, free scalar field on Minkowski spacetime $M = \mathbb{R}^{3,1}$ has phase space $(J^1 E)^* = \mathbb{R}_{x\mu}^3 \times \mathbb{R}_\phi \times \mathbb{R}_{\pi\mu}^3$ and Lagrangian

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The equations of motion are $(\partial_\mu \partial^\mu - m^2) \phi = 0$, and we have

$$\theta^\mu = \partial^\mu \phi \delta \phi = \pi^\mu \delta \phi \implies \omega^\mu = \delta \theta^\mu = \delta \pi^\mu \wedge \delta \phi. \quad (4.11)$$
The Noether Current

Consider the generator of spacetime translations:

$$\delta_\nu \phi = \partial_\nu \phi = \pi_\nu \implies \delta_\nu (\partial_\mu \phi) = \partial_\nu \partial_\mu \phi = \partial_\nu \pi_\mu.$$  (4.12)
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The corresponding variation in $L$ is

$$\delta_\nu L = \left[ \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right] \varepsilon_M =$$

$$=- \left[ (m^2 \phi) (\partial_\nu \phi) + (\partial^\mu \phi) (\partial_\nu \partial_\mu \phi) \right] \varepsilon_M =$$

$$=- \partial_\mu \left[ \frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) + \frac{1}{2} m^2 \phi^2 \right] \varepsilon_M = (\partial_\nu L) \varepsilon_M.$$  \hspace{1cm} (4.13)
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&= - \left[ (m^2 \phi) (\partial_\nu \phi) + (\partial^{\mu} \phi) (\partial_\nu \partial_\mu \phi) \right] \varepsilon_M = \\
&= - \partial_\mu \left[ \frac{1}{2} (\partial^{\mu} \phi) (\partial_\mu \phi) + \frac{1}{2} m^2 \phi^2 \right] \varepsilon_M = (\partial_\nu L) \varepsilon_M. \quad (4.13)
\end{align*}
\]

The conserved current is evidently the stress tensor:

\[
j^\mu_\nu = \theta^\mu_\nu - \sigma^\mu_\nu = \partial^{\mu} \phi \partial_\nu \phi - \delta^\mu_\nu L = T^\mu_\nu. \quad (4.14)
\]
Removing Gauge Symmetries

If two nearby field configurations $\phi$ and $\phi + \delta\phi$ represent the same physical state, then the vector $Z \sim \delta\phi$ is a degenerate direction in $\text{pre}$-phase space $\tilde{P}$, and the $\text{pre}$-symplectic form $\tilde{\Omega}$ is degenerate.

(More precisely, $\delta\phi = \mathcal{L}_Z \phi$, where $\mathcal{L}$ is the Lie derivative.)
If two nearby field configurations $\phi$ and $\phi + \delta\phi$ represent the same physical state, then the vector $Z \equiv \delta\phi$ is a degenerate direction in pre-phase space $\widetilde{P}$, and the pre-symplectic form $\widetilde{\Omega}$ is degenerate.

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Hamiltonian evolution by $Z$ is fake: $\iota_Z \widetilde{\Omega} = 0$.

Such $Z$ are zero modes of the pre-symplectic form $\widetilde{\Omega}$. The group of diffeomorphisms of $\widetilde{P}$ generated by $\ker \widetilde{\Omega}$ is the gauge group $G$. 
If two nearby field configurations $\phi$ and $\phi + \delta \phi$ represent the same physical state, then the vector $Z = \delta \phi$ is a degenerate direction in pre-phase space $\tilde{\mathcal{P}}$, and the pre-symplectic form $\tilde{\Omega}$ is degenerate.

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Such $Z$ are zero modes of the pre-symplectic form $\tilde{\Omega}$. The group of diffeomorphisms of $\tilde{\mathcal{P}}$ generated by $\ker \tilde{\Omega}$ is the gauge group $G$.

We formally glue all equivalent field configurations along all $Z$ to obtain the phase space $\mathcal{P} = \tilde{\mathcal{P}}/G$. We also obtain the symplectic form $\Omega = \tilde{\Omega}/G$ by gluing vector fields that differ by a zero mode.
In practice, one specifies $H$ and uses $\Omega$ to compute $X$, which effects phase space evolution. Since we have $\Omega$, we are “done”.
Covariance and Symmetry

In practice, one specifies $\mathcal{H}$ and uses $\Omega$ to compute $X$, which effects phase space evolution. Since we have $\Omega$, we are “done”.

- If $S[q(t), \dot{q}(t)]$ is invariant under $t \rightarrow t + \varepsilon$, the induced transformation $q(t) \rightarrow q(t + \varepsilon)$ on phase space is generated by $\delta q = q(t + \varepsilon) - q(t) = \varepsilon \dot{q}$ and has Noether charge $\mathcal{H}$.

- If $S[\phi(x), \partial_\mu \phi]$ is invariant under $x \rightarrow x + \varepsilon$, the induced phase-space symmetry $\delta_\mu \phi = \varepsilon \partial_\mu \phi$ yields the eigenvalues $\mathcal{H}^\mu$ of the stress tensor $T^{\mu\nu}$ as Noether charges.

- More generally, any transformation $x \rightarrow x'$ that generates a symmetry $\delta_\xi \phi$ has a corresponding Noether charge.
But in GR, coordinate transformations on $M$ are gauged and yield vanishing Noether charges, except when $\delta g$ arises from an isometry.

**Thus we ask:** how do we obtain a Hamiltonian vector field and its Noether charge for symmetries of $\mathcal{P}$ generated by isometries of $M$?
But in GR, coordinate transformations on $M$ are gauged and yield vanishing Noether charges, except when $\delta g$ arises from an isometry.

**Thus we ask:** how do we obtain a Hamiltonian vector field and its Noether charge for symmetries of $\mathcal{P}$ generated by isometries of $M$?

**The answer:** if $\xi$ is an isometry, the Hamiltonian vector field is

$$X_\xi = \left( \int_M \mathcal{L}_\xi \phi^a \right) \frac{\delta}{\delta \phi^a} = \left( \int_M \delta \xi \phi^a \right) \frac{\delta}{\delta \phi^a} \in T\mathcal{P}. \quad (4.15)$$

This vector field implements the flow of $\xi$ only on the dynamical fields $\phi$ in $\mathcal{P}$, and does not flow the rest of the gunk in spacetime.
The Noether current for \( X_\xi \) is essentially just \( \theta - \sigma \). More precisely, 
\[
J_\xi = \iota_{X_\xi} \theta - \nu_\xi. \quad \text{(This is a souped-up version of } H - p\dot{q} - L). 
\]
Finally, we seek the “Hamiltonian” \( H_\xi \) for which \( \iota_{X_\xi} \Omega = -\delta H_\xi \).
The Noether current for $X_\xi$ is essentially just $\theta - \sigma$. More precisely, $J_\xi = \iota_{X_\xi} \theta - \iota_{\xi}$. (This is a souped-up version of $H - p\dot{q} - L$.)

Finally, we seek the “Hamiltonian” $H_\xi$ for which $\iota_{X_\xi} \Omega = -\delta H_\xi$.

To find it, we use the explicit forms of $X_\xi$ and $\Omega$ to compute $\iota_{X_\xi} \Omega$. If the result is $\delta(\bigcirc)$, then “$\bigcirc$” is our $H_\xi$. Indeed,

$$H_\xi = \int_\Sigma J_\xi + \int_{\partial \Sigma} (\iota_\xi \delta \ell - \iota_{X_\xi} \mathcal{O}),$$

(4.16)

where $\ell$ is (!) the Lagrangian on $\partial M$. 
Outline

1. The Problem of Time

2. Particle Mechanics
   - The Variational Principle
   - Hamiltonian Mechanics

3. Particles and Fields

4. Covariant Phase Space
   - Theme and First Variation
   - Example: Free Scalar
   - Diffeomorphism Charges

5. Gravity at Last
The Action and its Variation

Let \((M, g)\) have boundary \((\partial M, \gamma)\). The full gravity action consists of the **Einstein-Hilbert** and **Gibbons-Hawking-York** terms:

\[
S = S_{EH} + S_{GHY} = \int_M L + \int_{\partial M} \ell =
\]

\[
= \frac{1}{16\pi G} \int_M R \varepsilon_M + \frac{1}{8\pi G} \int_{\partial M} K \varepsilon_{\partial M}.
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\]

The variation of \(L\) leads to the Einstein field equations:

\[
\delta L = \mathcal{E}^{\mu\nu} \delta g_{\mu\nu} + d\Theta, \quad \mathcal{E}^{\mu\nu} = \frac{1}{16\pi G} \left( -R^{\mu\nu} + \frac{1}{2} R g^{\mu\nu} \right) \epsilon_M. \tag{5.2}
\]

Meanwhile, \(\delta \ell = \frac{1}{16\pi G} (\text{stuff}) \epsilon_{\partial M}\) contributes to \(\Theta\) on \(\partial M\).
After a short calculation, we obtain

\[
(\Theta + \delta \ell) \bigg|_{\partial M} = -\frac{1}{16\pi G} (K^{\mu\nu} - K^\gamma_{\mu\nu}) \delta g_{\mu\nu} \varepsilon_{\partial M} + dC =
\]

\[
= \frac{1}{2} T_{BY}^{\mu\nu} \delta g_{\mu\nu} \varepsilon_{\partial M} + dC,
\]

\[
C = -\frac{\gamma^{\mu\nu} n^\alpha \delta g_{\nu\alpha}}{16\pi G} \cdot \varepsilon_{\partial M} \neq 0
\]

(5.3)
The Symplectic Potential and Form

After a short calculation, we obtain

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\]

(5.3)

The boundary-corrected symplectic potential in GR consists of the Brown-York stress tensor and another total divergence.

Taking \( \delta g_{\mu\nu} \bigg|_{\partial M} = 0 \), i.e. fixing the metric on \( \partial M \), does not set the boundary term in \( \delta S \) to zero! See [Harlow-Wu 2019] for an explanation of why such terms should generally be present.
Once we allow for a nonzero flux from $dC$ in the on-shell variation

$$\delta S = \int_{\partial M} (\Theta + \delta \ell) = \int_{\partial M} \left( \frac{1}{2} T_{BY}^{\mu\nu} \delta g_{\mu\nu} + dC \right) \varepsilon_{\partial M}, \quad (5.4)$$

the Dirichlet boundary conditions $\delta \gamma = 0$ render the variational problem in GR well-posed in a covariant way. Viewing $\gamma$ as a fixed source reminds one of the extrapolate dictionary in AdS/CFT.
Once we allow for a nonzero flux from $dC$ in the on-shell variation

$$\delta S = \int_{\partial M} (\Theta + \delta \ell) = \int_{\partial M} \left( \frac{1}{2} T_{\mu \nu}^{BY} \delta g_{\mu \nu} + dC \right) \epsilon_{\partial M},$$

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the Dirichlet boundary conditions $\delta \gamma = 0$ render the variational problem in GR well-posed in a covariant way. Viewing $\gamma$ as a fixed source reminds one of the *extrapolate dictionary* in AdS/CFT.

By Stokes’s theorem, the boundary-of-a-boundary term $C$ lives on the codimension-2 *corners* of the spacetime. Holography, anyone?

There are also lines of research investigating “edge modes” and “corner potentials” in gravity that are somewhat related.
Diffeomorphism Charges

After some inspiration by Wald and a straightforward calculation of Harlow-Wu, one finds the diffeomorphism charges of GR:

\[ J_\xi = dQ_\xi \implies H_\xi = - \int_{\partial \Sigma} \tau^\mu \xi^\nu T^{BY}_{\mu \nu} \varepsilon_{\partial \Sigma}. \quad (5.5) \]

This is the expression for the generators of boundary isometries with Killing field \( \xi^\mu \), and is once again a corner term.
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**Commentary:** CPS is powerful and recovers hard results in GR (ADM, BY, even \( S_{BH} \)) with relative ease. It smells a lot like holography ("AdS/CFT is just spicy Stokes’s theorem"), and is way too beautiful *not* to be immediately adopted by everyone.
Summary and Conclusions

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