An Introduction to Elliptic Functions

A Walking Tour of the Elliptic Zoo

David Grabovsky

Berenstein High Energy Theory and Gravity Group

May 18, 2021

An Introduction to ~~Elliptic Functions~~

A ~~Walking~~ Tour of the Elliptic ~~Zoo~~

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An Introduction to Abelian Integrals

A Safari Tour of the Elliptic Prairie

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Outline

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Holomorphic Functions

Let $\Omega \subset \mathbb{C}$ be an open, connected subset (i.e. a **domain**) of $\mathbb{C}$.
We set $i^2 = -1$ and write $z = x + iy \in \mathbb{C}$ for a complex number.

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A function $f \colon \Omega \longrightarrow \mathbb{C}$ is **holomorphic** or **analytic** on Ω if

$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z} \equiv \lim_{z \to h} \left(\frac{f(z+h) - f(z)}{h} \right) \tag{1.1}$$

exists for all $z \in \Omega$.

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Complex differentiability implies the **Cauchy-Riemann equations**,

$$\frac{\partial f}{\partial \bar{z}} \equiv 0. \tag{1.2}$$

It turns out that complex differentiability is an extremely strong property.

Holomorphic Miracles

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The following are equivalent to the holomorphicity of $f : \Omega \longrightarrow \mathbb{C}$ *on* Ω*:*

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- *For every disk* $D_r \subset \Omega$ *and every* $z \in D_r$, f *satisfies*

$$f(z) = \frac{1}{2\pi i}\oint_{\partial D} \frac{f(w)}{z-w}\,\mathrm{d}w \qquad \text{(Cauchy integral formula)}. \tag{1.4}$$

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- *For every* $z_0 \in \Omega$*, there exists a disk* $D_r(z_0) \subset \Omega$ *such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad \text{(\textbf{Taylor series})} \tag{1.5}$$

converges uniformly for all $z \in D_r(z_0)$*. In particular,* $f \in C^{\omega}(\Omega)$*.*

Local Structure Near Zeros

Let $f\colon \Omega \longrightarrow \mathbb{C}$ be a holomorphic function.

Proposition (Zeros are isolated)

If f is not identically zero, then the zeros of f are isolated: if $f(z_0) = 0$, then there is a neighborhood U of z_0 where $f(z) \neq 0$ for all $z \in U \setminus \{z_0\}$.

Proof.

Expand $f = \sum_n c_n(z - z_0)^n$. Since $f \neq 0$, consider the smallest N for which $c_N \neq 0$. Write $f(z) = (z - z_0)^N g(z)$, where $g(z_0) = a_N \neq 0$. □

N.B. The **order** of a zero of f at z_0 is N.

Theorem (Local n-fold covering)

Let f have a zero of order n at z_0. Then for all v near $0 \in \mathbb{C}$, there are n points $\{z_i\}$ near $z_0 \in \Omega$ such that $f(z_1) = \cdots f(z_n) = v$.

Open Mapping and Maximum Modulus

Theorem (Open mapping)

If $f : \Omega \longrightarrow \mathbb{C}$ is a non-constant holomorphic function, then f maps open sets to open sets. In particular, $f(\Omega) \subset \mathbb{C}$ is open.

"Non-constant holomorphic maps are open."

Theorem (Maximum modulus principle)

If $f : \Omega \longrightarrow \mathbb{C}$ is a non-constant holomorphic function, then there cannot exist $z_0 \in \Omega$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \Omega$.

"The maximum of $|f|$ can only be attained on $\partial\Omega$."

Liouville, Picard, and More

Definition (Entire function)

A function holomorphic on the whole complex plane is called **entire**.

Theorem (Liouville)

Every bounded entire function is constant.

Liouville kills off interesting global extensions of holomorphic maps.

Corollary (Functions on $\mathbb{P}^1$)

Every holomorphic function $f : \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C}$ *is constant.*

Theorem (Little Picard)

If $f : \mathbb{C} \longrightarrow \mathbb{C}$ *is entire and non-constant, then the image of* f *is either the whole complex plane or the plane minus a single point.*

Meromorphic Functions

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Definition (Meromorphic function)

A function $g\colon \Omega \longrightarrow \mathbb{C}$ is **meromorphic** on Ω if, for every $z_0 \in \Omega$, $g(z)$ can be expressed in terms of its **Laurent series**:

$$g(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n, \qquad N \in \mathbb{Z}. \tag{1.6}$$

If $N < 0$, z_0 is a **pole** of g of **order** $|N|$. If $N = -1$, the pole is **simple**. If z_0 is a pole of g, the **residue** of g at z_0 is $\mathrm{Res}[g;\, z_0] \equiv a_{-1}$.

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Theorem (Residue)

If $g\colon \Omega \longrightarrow \mathbb{C}$ is meromorphic with poles $z_1, \ldots, z_k$ inside $D \subset \Omega$, then

$$\frac{1}{2\pi i} \oint_{\partial D} g(z)\, \mathrm{d}z = \sum_{i=1}^{k} \operatorname{Res}[g;\, z_i]. \tag{1.7}$$

The View from Geometry

"Proof" (residue theorem).

All terms $(z - z_0)^n$ in the Laurent series of g are total derivatives except for $n = -1$. Aside from this term, $\widetilde{g}(z)\,\mathrm{d}z = \mathrm{d}\omega$ is exact, so

$$\oint_{\partial D} \widetilde{g}(z)\,\mathrm{d}z = \oint_{\partial D} \mathrm{d}\omega = \int_{\partial(\partial D)} \omega = \int_{\emptyset} \omega = 0. \tag{1.8}$$

But the form $\frac{\mathrm{d}z}{z-z_0}$ is *not* exact! Therefore **residues measure topology**; in particular, they generate $H^1_{\mathrm{dR}}(D) \cong H_1(D)$ and "algebrize" ∂D. □

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Extend the range of g from $\mathbb{C}$ to $\mathbb{P}^1$, allowing $g(z_0) = \infty$. This turns poles into regular points, so **meromorphic functions are holomorphic maps to $\mathbb{P}^1$.** This is "obvious," since poles of g are zeros of $\frac{1}{g}$.

Q: Can we extend the *domain* of a complex function beyond $\mathbb{C}$ as well?

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The Square Root

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Write $z = re^{i\theta} \implies \sqrt{z} = \sqrt{r}\, e^{i\theta/2}$. On the upper and lower sides of $\mathbb{R}_{\geq 0} \subset \mathbb{C}$, the angle θ jumps from $0 + \varepsilon$ to $2\pi - \varepsilon$. As $\varepsilon \longrightarrow 0$, we get

$$\begin{cases} \sqrt{z_+} \sim \sqrt{r}\, e^0 & = +\sqrt{r}\,, \\ \sqrt{z_-} \sim \sqrt{r}\, e^{2\pi i/2} & = -\sqrt{r}\,. \end{cases} \qquad \text{Oh no...} \qquad (2.1)$$

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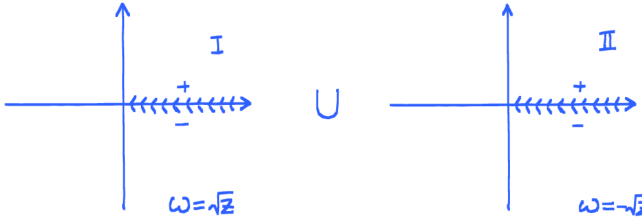
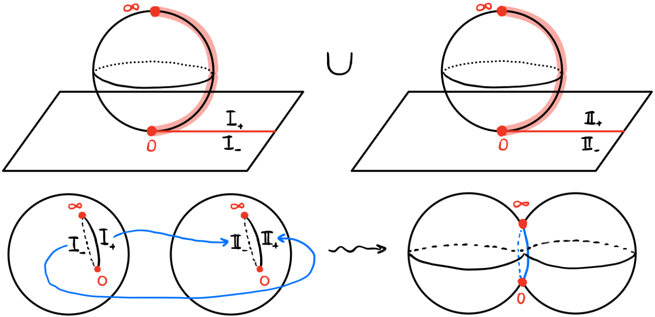
Solution: glue together two copies of $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$: I_+ to II_- and I_- to II_+.



The Riemann Sphere: Construction

We obtain a surface $X = \mathrm{I} \cup \mathrm{II} \simeq S^2 \setminus \{N, S\}$:



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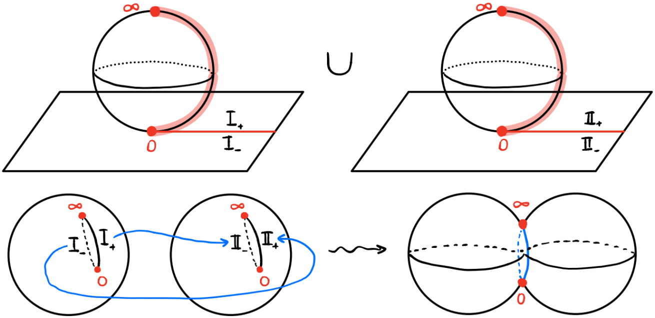


Then the generalized square root w is continuous on X:

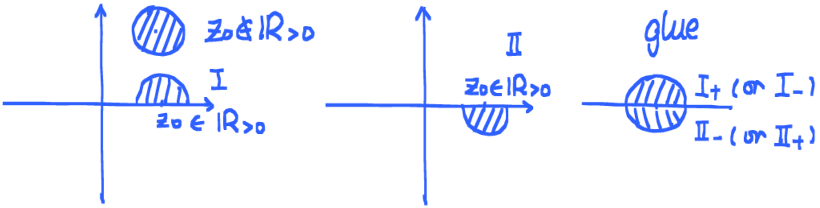
$$w = \begin{cases} +\sqrt{z}, & z \in \mathrm{I}, \\ -\sqrt{z}, & z \in \mathrm{II}. \end{cases} \tag{2.2}$$

Claim: The holes on X can be plugged, and w is holomorphic on $\hat{X}$.

The Riemann Sphere: Plugging the Holes

Proposition (Plugging the holes)

Every $z_0 \in X$ admits a neighborhood $U_{z_0} \subset X$ biholomorphic to a disk $D \subset \mathbb{C}$. *The same is true of* $0 = S \notin X$ *and* $\infty = N \notin X$.



Proof (sketch).

If z_0 is not on a cut, take $U_{z_0} = D_\varepsilon(z_0)$. If it is, glue two half-disks together. Take the holomorphic coordinate $z \mapsto t = z - z_0$. □

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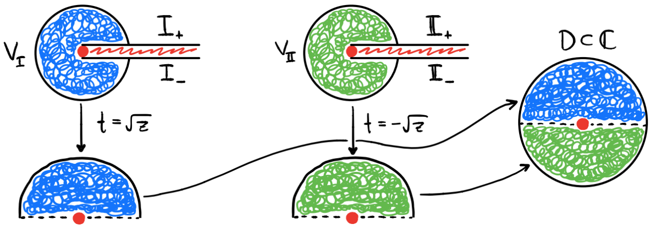

Proof (sketch).

Near 0, glue together two cut disks in I and II, and take $z \mapsto t = \pm\sqrt{z}$.
Near ∞, glue the exterior regions of disks and use $t = \pm\frac{1}{\sqrt{z}}$. □

Holomorphicity of w

A **Riemann surface** $\mathcal{R}$ is a connected, complex 2-manifold.

Definition (Holomorphic maps on Riemann surfaces)

A function $f\colon \mathcal{R} \longrightarrow \mathbb{C}$ is **holomorphic** at $p \in \mathcal{R}$ if it is holomorphic in any coordinate chart (U, z) of p, i.e. if $f\big|_U(z(t))$ is holomorphic in t.

The map $w = \sqrt{z}$, defined on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, was extended to a continuous map $w\colon X \longrightarrow \mathbb{C}$. Holomorphic coordinates were put on $X \cup \{S, N\}$.

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Proposition (Maximal extension of the square root)

The map w is holomorphic on X. Moreover, by defining $w(0) = 0$ and $w(\infty) = \infty$, we obtain a meromorphic function $w\colon \hat{X} \longrightarrow \mathbb{C}$ on the two-sheeted **Riemann sphere** $\hat{X} = \mathrm{I} \cup \mathrm{II} \cup \{S, N\} = \mathbb{P}^1 \simeq S^2$.

In local coordinates $z_0(t) \sim t^2$ and $z_\infty(t) \sim \frac{1}{t^2}$, the function w has a simple zero at 0 and a simple pole at ∞, and w is odd under $\mathrm{I} \longleftrightarrow \mathrm{II}$.

A More Interesting Function

Consider the function w given by $w^2 = z(z-1)(z-\lambda)$, with $\lambda \notin \{0,1\}$.

What is the Riemann surface defined by w? To extend w maximally, it suffices that $\sqrt{z}$, $\sqrt{z-1}$, and $\sqrt{z-\lambda}$ all be simultanously well defined.

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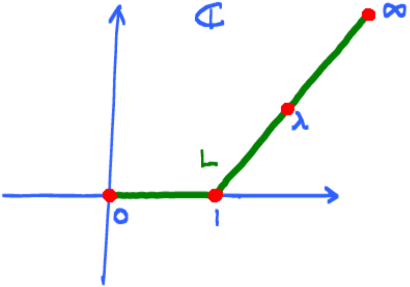
Let L be a curve through $0, 1, \lambda, \infty$. Clearly w is well defined on $\mathbb{C} \setminus L$.



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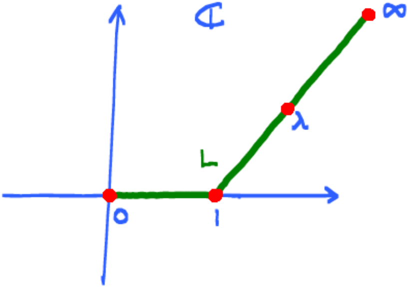

Strategy: for each segment of L, follow the factors $\sqrt{z}$, $\sqrt{z-1}$, $\sqrt{z-\lambda}$ around a curve piercing that segment, and count up the minus signs.

Pick $z \in [0,1] = L_{01}$ and follow C:

$$\begin{cases} \sqrt{z} & \overset{C}{\longmapsto} -\sqrt{z}, \\ \sqrt{z-1} & \overset{C}{\longmapsto} +\sqrt{z-1}, \\ \sqrt{z-\lambda} & \overset{C}{\longmapsto} +\sqrt{z-\lambda}. \end{cases} \tag{2.3}$$

Next, pick $z \in L_{1\lambda}$ and follow D:

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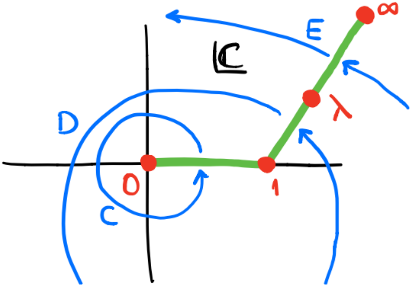
Finally, pick $z \in L_{\lambda\infty}$ and follow E:

$$\begin{cases} \sqrt{z} & \overset{E}{\longmapsto} -\sqrt{z}, \\ \sqrt{z-1} & \overset{E}{\longmapsto} -\sqrt{z-1}, \\ \sqrt{z-\lambda} & \overset{E}{\longmapsto} -\sqrt{z-\lambda}. \end{cases} \tag{2.5}$$

Some Quick Calculations



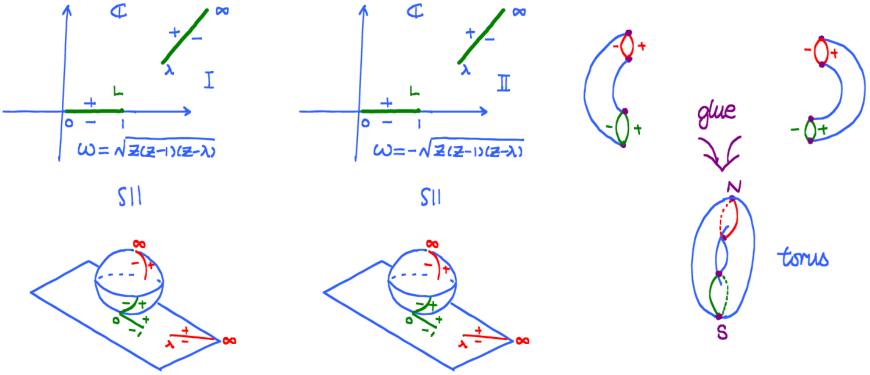
To summarize:

$$\begin{cases} w & \overset{C}{\longmapsto} -w, \\ w & \overset{D}{\longmapsto} +w, \\ w & \overset{E}{\longmapsto} -w. \end{cases} \tag{2.6}$$

So w is only singular on $[0,1] \cup L_{\lambda\infty}$.

The Complex Torus: Construction

We perform the same gluing procedure as before to get $X = \mathrm{I} \cup \mathrm{II}$:



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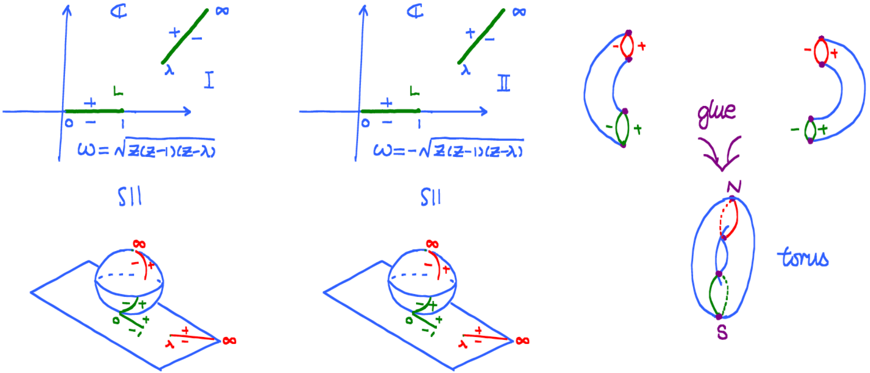

As before, we can find holomorphic coordinates near $0, 1, \lambda, \infty$ to plug the holes and obtain the **torus** $\hat{X} = X \cup \{0, 1, \lambda, \infty\} \simeq T^2$.

Holomorphicity of w

Proposition (Maximal extension to the torus)

The map $w = \pm\sqrt{z(z-1)(z-\lambda)}$ *is meromorphic on* $\hat{X}$. *It has three simple zeros at* 0, 1, *and* λ, *and a pole of order* 3 *at* ∞.

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Proof.

Away from 0, 1, λ, and ∞, w is clearly holomorphic.

Near 0, take $t = \pm\sqrt{z}$; then, $w = \pm t\sqrt{(t^2-1)(t^2-\lambda)}$ has a simple zero at 0. The same is true for $t \sim \sqrt{z-1}$ and $t \sim \sqrt{z-\lambda}$ at 1 and λ.

Finally, near ∞, take $t = \pm\frac{1}{\sqrt{z}}$; then, $w = \frac{1}{t^3}\sqrt{(1-t^2)(1-\lambda t^2)}$ has a pole of order 3 at ∞. Notice the resemblance to elliptic integrals! □

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Big question: There are many tori, $T^2_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, characterized by a complex **modulus** $\tau \in \mathbb{H}^2_+$. Which one is $\hat{X}$? How does λ determine τ?

Outline

Functions on the Torus

Definition (Classification of functions on the torus)

Every function $f : T^2_\tau \longrightarrow \mathbb{C}$ must be **doubly periodic** on $\mathbb{C}$, i.e. $f(z+1) = f(z) = f(z+\tau)$.

- A holomorphic doubly periodic function on $\mathbb{C}$ is constant.
- A meromorphic doubly periodic function on $\mathbb{C}$ is **elliptic**.

Fix constants $\eta_1, \eta_2 \in \mathbb{C}$. Any function $f : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying $f(z+1) - f(z) = \eta_1$ and $f(z+\tau) - f(z) = \eta_2$ is **quasiperiodic**.

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So $\hat{X}$ supports only constant holomorphic functions. However, *differences* of quasiperiodic functions can be used to build **holomorphic forms** on $\hat{X}$.

Strategy: (1) construct holomorphic forms explicitly; (2) develop techniques for manipulating them; (3) obtain a map $\hat{X} \longleftrightarrow T^2_\tau$.

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Proposition (A miracle)

The form $\omega = \frac{\mathrm{d}z}{w}$ *is globally holomorphic and nowhere vanishing on* $\hat{X}$.

Forms are Better Than Functions

Proof (ω is holomorphic).

We check holomorphicity by expressing ω explicitly in local coordinates.

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- Away from $0, 1, \lambda, \infty$, $t = z$ does the job.
- Near 0, $t = \pm\sqrt{z} \iff z = t^2$ implies $\mathrm{d}z = 2t\,\mathrm{d}t$. Then,

$$\omega = \frac{\mathrm{d}z}{w} = \frac{2t\,\mathrm{d}t}{\sqrt{t^2(t^2-1)(t^2-\lambda)}} = \frac{2\,\mathrm{d}t}{\sqrt{(t^2-1)(t^2-\lambda)}}. \tag{3.1}$$

- The same works near 1 and λ, with $t_1 = \sqrt{z-1}$ and $t_\lambda = \sqrt{z-\lambda}$.

Forms are Better Than Functions

Proof (ω is holomorphic).

We check holomorphicity by expressing ω explicitly in local coordinates.

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- Near ∞, $t = \pm\frac{1}{\sqrt{z}} \iff z = \frac{1}{t^2}$ implies $\mathrm{d}z = -\frac{2\,\mathrm{d}t}{t^3}$. Then,

$$\omega = \frac{\mathrm{d}z}{w} = \frac{-2\,\mathrm{d}t/t^2}{\sqrt{\frac{1}{t^2}\left(\frac{1}{t^2}-1\right)\left(\frac{1}{t^2}-\lambda\right)}} = \frac{-2\,\mathrm{d}t}{\sqrt{(1-t^2)(1-\lambda t^2)}}. \qquad (3.2)$$

In each case, ω is locally holomorphic and nonvanishing. (!!!) □

The Abel-Jacobi Map

Definition ($\widetilde{\text{A}}$bel map)

Fix a point $p_0 \in \hat{X}$. The **$\widetilde{\text{A}}$bel map** $\widetilde{A}\colon \hat{X} \longrightarrow \mathbb{C}$ is given by

$$p \mapsto \widetilde{A}(p) = \int_{p_0}^{p} \omega = \int_{p_0}^{p} \frac{\mathrm{d}z}{w}, \tag{3.3}$$

integrated along a curve $\gamma \subset \hat{X}$ from p_0 to p.

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Since ω is holomorphic, $\widetilde{A}(p)$ is a homotopy invariant, so it depends only on $[\gamma]$. If we choose independent cycles A, B in $\hat{X}$, then $\widetilde{A}$ descends to

$$\widetilde{A}(p) \in \mathbb{C}/(\alpha\mathbb{Z} \oplus \beta\mathbb{Z}), \qquad \alpha = \oint_A \omega, \qquad \beta = \oint_B \omega. \tag{3.4}$$

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In fact $\alpha, \beta \neq 0$, so we normalize ω by setting $\hat{\omega} = \frac{\omega}{\alpha}$, or choose A so that

$$\oint_A \omega = 1, \qquad \oint_B \omega \equiv \tau \in \mathbb{C}. \tag{3.5}$$

The Jacobi Inversion Theorem

Definition (The Abel map)

Fix a point $p_0 \in \hat{X}$. With conventions as above, the **Abel map** is

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The modulus is a **period integral**. In local coordinates near ∞, $A(p)$ is an **elliptic integral** of the first kind, and $A^{-1}(p)$ is an **elliptic function**.

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For injectivity ($p \neq q \implies A(p) \neq A(q)$), suppose that $A(p) = A(q)$ for $p \neq q$. **Abel's theorem** says that $A(p) = A(q)$ iff there is a meromorphic function f on $\hat{X}$ with a simple zero at p and a simple pole at q.

But then consider the **meromorphic form** $\psi = f \frac{\mathrm{d}z}{w} = f\omega$. Since ω is nonvanishing, ψ has a simple pole at q and thus a nonzero residue there. But this cannot be, since every meromorphic form ψ must satisfy

$$\sum_{\text{poles } p_i} \operatorname{Res}[\psi; p_i] = 0. \qquad \square$$

Abel's Theorem

Theorem (Abel)

Let $p_1, \ldots, p_M$ and $q_1, \ldots, q_N$ be points of $\hat{X}$, counted with multiplicity. There exists a meromorphic function $f\colon \hat{X} \longrightarrow C$, with zeros at the p_i and poles at the q_j, if and only if $M = N$ and

$$\sum_{i=1}^{M} A(p_i) = \sum_{j=1}^{N} A(q_j), \tag{3.8}$$

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How to construct f? Elliptic functions with no poles are constants, and elliptic functions with a single pole do not exist $(\deg g = 0)$. So...

1. **Abel-Riemann:** use meromorphic forms ω_{pq} with two simple poles.
2. **Weierstraß:** build a meromorphic function $\wp$ with a double pole.
3. **Jacobi:** use modular invariance to construct ϑ functions.
4. **Hodge:** use harmonic forms, the $\bar{\partial}$ construction, and PDE.

Some Preparatory Work

Proof (Abel, $M = N$).

Iff ϕ is meromorphic on $\hat{X}$ with prescribed zeros and poles, $\eta = \frac{d\phi}{\phi}$ is meromorphic with poles at p_i *and* q_i. (Pf: Laurent and power rule.) Thus

$$\sum_{\text{poles}} \operatorname{Res}[\eta] = \sum_{i=1}^{M} \operatorname{Res}[\eta;\, p_i] + \sum_{j=1}^{N} \operatorname{Res}[\eta;\, q_j] = M - N \overset{!}{=} 0.$$

□

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Lemma (Existence of meromorphic forms)

For any two points $q_1, q_2 \in \hat{X}$, there is a meromorphic form $\omega_{q_1 q_2}$ with simple poles at q_1 and q_2 with residues $+1$ and -1, respectively. There is also a meromorphic form ω_{q_1} with a double pole at q_1.

Proof.

The proof gets technical, but the forms ω_{q_1} and $\omega_{q_1 q_2}$ may be constructed explicitly by outfitting $\omega = \frac{\mathrm{d}z}{w}$ with singularities. $\square$

Construction of Elliptic Functions

Assuming the lemma, fix $p_0 \in \hat{X}$ and consider the meromorphic form

$$\psi = \sum_{i=1}^{N} \omega_{p_0 p_i} - \sum_{i=1}^{N} \omega_{p_0 q_i}. \tag{3.9}$$

The form ψ has simple poles at p_i (residue $+1$) and q_i (residue -1). All of the poles at p_0 get canceled between the two sums!

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Idea: Since ψ has the right pole structure, construct f by the ansatz $\psi \sim \frac{\mathrm{d}f}{f}$. This is *almost* well defined on $\hat{X}$; we must subtract off a multiple of ω, which does not affect the poles. One may then recover f:

$$\psi - c\omega = \frac{\mathrm{d}f}{f} \implies f \text{ “}=\text{” } \exp\left[\int \frac{\mathrm{d}f}{f}\right] = \exp\left[\int (\psi - c\omega)\right]. \tag{3.10}$$

The “ $=$ ” step is where $\sum_{i=1}^{N} [A(p_i) - A(q_i)] = 0$ becomes necessary and sufficient. The key tool is a local version of the Abel map on $\mathbb{C}$.

Outline

Where We've Been and Where We're Headed

What we have accomplished so far:

1. Reviewed complex analysis from the viewpoint of differential forms.
2. Constructed the Riemann surface $\mathbb{P}^1$ for the square root.
3. Constructed the Riemann surface $\hat{X}$ for a cubic equation.
4. Discovered a holomorphic form on $\hat{X}$ and integrated it on a basis of cycles of $\hat{X}$ to define the Abel map $A\colon \hat{X} \longleftrightarrow T^2_\rho$.
5. Proved that this map is holomorphic and bijective, in the process constructing meromorphic forms and elliptic functions on $\hat{X}$.

What's next: do step 5 three more times, à la Weierstraß, Jacobi, and Hodge. The Abel approach was conceptually clean, but does not reveal the structure, symmetries, or spectral properties of elliptic functions.

Nel mezzo del cammin di nostra vita
Mi ritrovai per una selva oscura
Ché la diritta via era smarrita.

When I had journeyed half our life's way,
I found myself within a shadowed forest,
For I had lost the path that does not stray.