Self-Force and Radiation Reaction;

Or, How to Annoy Your 110 Professor

David Grabovsky

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The "resolution" is that point charges are pathological. Carefully taking the pointlike limit of an extended distribution yields finite self-effects.

Welcome to the Maison Électromagnétique de SPS!

On the Menu Today

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- **Sources:** charge distribution ρ and current $\mathbf{J}$. Collectively, J^μ.
- **Fields:** electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Collectively, $F_{\mu\nu}$.
- **Particles:** points $\mathbf{r}(t)$ of mass m and charge q. Collectively, x^μ.

Test Charges and Self-Consistency

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Warning: This contradicts textbooks, which demand that charges feel no self-forces. This is the heart of the problem, and will be taken up later.

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Q: Can we determine the self-force and solve for the particle's true path?
A: Yes, but you're not going to like the answer. Among the predictions are exponentially accelerating (**runaway**) and **retrocausal** solutions.

On the Menu Today

1. *Hors d'oeuvre:* Philosophy
2. *Entrée:* The Self-Force and its Problems
3. *Plat Principal:* To Infinity and Back
4. *Dessert:* Extensions and Conclusions

Derivation of the Abraham-Lorentz Force

The **Larmor formula** gives the power radiated by an oscillating charge:

$$P = \frac{\mu_0 q^2}{6\pi c}\mathbf{a}^2 = \frac{\mu_0 q^2}{6\pi c}(\mathbf{a}\cdot\mathbf{a}). \tag{2.1}$$

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The average work done on the particle by the radiation it emits over one period is caused by an **Abraham-Lorentz force** that pushes it back:

$$W = -\int_{t_1}^{t_2} P\,\mathrm{d}t = \int_{t_1}^{t_2} \mathbf{F}_{\mathrm{rad}}\cdot \mathrm{d}\mathbf{r} = \int_{t_1}^{t_2} \big(\mathbf{F}_{\mathrm{rad}}\cdot\mathbf{v}\big)\mathrm{d}t. \tag{2.2}$$

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We can compute W from the Larmor formula by integrating by parts:

$$W = -\frac{\mu_0 q^2}{6\pi c}\int_{t_1}^{t_2}\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\mathrm{d}t = \cancel{-\frac{\mu_0 q^2}{6\pi c}\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\cdot\mathbf{v}\right)\Bigg|_{t_1}^{t_2}} + \frac{\mu_0 q^2}{6\pi c}\int_{t_1}^{t_2}\left(\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2}\cdot\mathbf{v}\right)\mathrm{d}t$$

$$= \int_{t_1}^{t_2}\left(\frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}\cdot\mathbf{v}\right)\mathrm{d}t \stackrel{!}{=} \int_{t_1}^{t_2}(\mathbf{F}_{\mathrm{rad}}\cdot\mathbf{v})\mathrm{d}t \implies \boxed{\mathbf{F}_{\mathrm{rad}} = \frac{\mu_0 q^2}{6\pi c}\dot{\mathbf{a}}.} \tag{2.3}$$

The Problems Begin: Runaway Solutions

The RR force is *third-order* in time. Its initial value problem is ill-posed:

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Response 1 (cop-out): "This is a genuine prediction of EM, and signals its breakdown. The theory must be replaced by QED at small scales."

Response 2 (Dirac): "Eliminate runaway solutions by imposing $\mathbf{a}_0 \equiv \mathbf{0}$."

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But if $\mathbf{F}_{\text{ext}} \neq \mathbf{0}$, things are even worse. The EOM $m\ddot{\mathbf{r}} = m\tau\dddot{\mathbf{r}} + \mathbf{F}_{\text{ext}}(t)$ may be re-expressed in terms of $\mathbf{a} = \ddot{\mathbf{r}}$ and (formally) solved:

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Sadly, the **Compton wavelength** $\lambda_e \approx 10^{-12}$ m $\gg r_e$ makes quantum effects kick in far before pre-acceleration could become observable.

The Problems Escalate: An Example

Suppose that $\mathbf{F}_{\text{ext}} = \mathbf{F}_0$ is constant on $[0, T]$ and vanishes otherwise. The particle experiences both pre-acceleration and runaway behavior:

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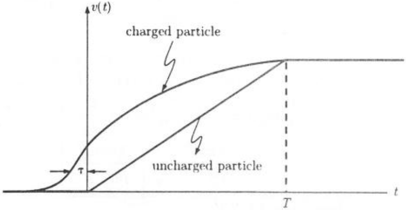
By a choice of $\mathbf{a}_0$, we can eliminate either pre-acceleration or runaway behavior, but not both: Lorentz-Abraham is *definitely* pathological.



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The most striking consequence of the self-force is that it vanishes when $\mathbf{a} = \mathbf{a}_0$ is constant. In *free fall*, $\mathbf{F}_{\mathrm{rad}} = \mathbf{0}$, and yet power is radiated:

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- **Blaes:** Thinking about forces is misguided. One can find the energy lost to radiation, and conserved quantities determine the motion.
- **Feynman:** Uniform acceleration is not periodic, so our derivation of $\mathbf{F}_{\text{rad}}$ picks up a boundary term. The radiation should live there.
- **Einstein:** The Equivalence principle views constant acceleration as *no* acceleration. So a freely falling point charge should not radiate!
- **Wald:** The issue is deeply related to the particle's self-energy.

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Griffiths: "Consider the force exerted by a particle on itself."

You: w h a t

Griffiths: "So causality and the conservation of energy are violated, and the electron has an infinite self-energy, which counts this time."

You:

Griffiths: "This is because point charges fundamentally don't exist."

Renormalizing Away the Problem

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Lorentz warrants attention, and Dirac was on the right track. (haha)

N.B. We are now fully relativistic. Mass is energy, space is time, and the self-force generalizes to the **Abraham-Lorentz-Dirac (ALD)** equation.

The Resolution at Last

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Upshot: EM is not broken, nor does QM need to repair it. There are no infinities in the theory, nor is there any violation of physical principles.

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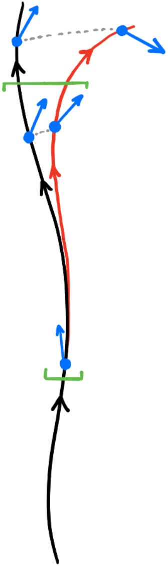
If $\mathbf{a}$ on the RHS is the *true* acceleration, then the ALD is third-order. But if we use its *background* value, then $\mathbf{a} \sim \mathbf{v}$, so $\dot{\mathbf{a}}$ is only second-order in time!



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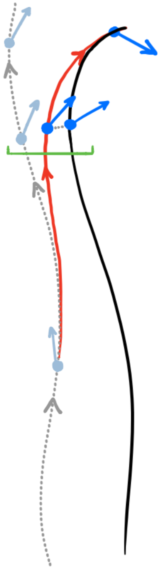
Small deviations might pile up and cause this to become a poor approximation. This is okay, as long as we *also* re-compute the Lorentz worldline as the particle moves.



On the Menu Today

The Gravitational Setting

GR is a field theory: the sources (all known mass-energy) determine the fields (the spacetime metric), which in turn give the sources dynamics. *Matter tells spacetime how to curve; spacetime tells matter how to move.*

But the gravitational field carries energy, so it is also a source: gravity gravitates. This makes the Einstein field equations (EFE) nonlinear.

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Gralla–Wald (2011) have attacked this problem too.

In "real life," self-forces in GR are important in GW and BH simulations.

Quantum Bells and Whistles

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I am also told that there is a quantum version of the ALD equation.

Feynman-Wheeler absorber theory, where electrons both emit and absorb $\mathbb{E}$-fields, postulates that particles do *not* interact with themselves. It gives a causal description of the ALD force, but not without issues.

TL;DR

- In EM, we specify the sources ρ and $\mathbf{J}$, determine the $\mathbf{E}$ and $\mathbf{B}$ fields, and find the motion of a test charge in these external fields.
- But test charges produce their own fields, and their self-influence should modify their trajectory in a self-consistent manner.

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- For example, the radiation emitted by an accelerated charge produces an Abraham-Lorentz damping force proportional to $\dot{\mathbf{a}}$.
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- The issue of uniform acceleration has inspired much debate.
- Proposed resolutions focus on taking the pointlike limit of a finite-size electron and renormalizing away the infinite self-energy.
- Wald takes the limit carefully and gets a finite self-energy. Replacing $\mathbf{a}$ by its background value makes the ALD equation pathology-free.
- Bells and whistles in GR and QM abound, and much work remains!

Further Reading

- Wikipedia, "Abraham-Lorentz Force," https://en.wikipedia.org/wiki/Abraham-Lorentz_force.
- Prof. Omer Blaes, 210B lecture notes.
- Griffiths, "Introduction to Electrodynamics," §11.2.
- "Does a Uniformly Accelerating Charge Radiate?" https://www.mathpages.com/home/kmath528/kmath528.htm.
- McDonald, "On the History of the Radiation Reaction," https://www.physics.princeton.edu/~mcdonald/examples/selfforce.pdf.
- Dirac, "Classical Theory of Radiating Electrons," https://royalsocietypublishing.org/doi/pdf/10.1098/rspa.1938.0124.
- Gralla, Harte, Wald, "A Rigorous Derivation of Electromagnetic Self-Force," https://arxiv.org/pdf/0905.2391.pdf.

Just for Fun

Here's what you can get away with if it's 1938 and you're Dirac:

The object of the paper is to set up in the classical theory a self-consistent scheme of equations which may be used to calculate all the results that can be obtained from experiment about the interaction of electrons and radiation. The electron is treated as a point charge and the difficulties of the infinite Coulomb energy are avoided by a procedure of direct omission or subtraction of unwanted terms, somewhat similar to what has been used in the theory of the positron. The equations obtained are of the same form as those already in current use, but in their physical interpretation the finite size of the electron reappears in a new sense, the interior of the electron being a region of space through which signals can be transmitted faster than light.