FM Spectroscopy

Or, How Excessive Trigonometry Solves All Problems

David Grabovsky

Columbia University

July 3, 2017

Laser Locking for 5-Year-Olds

Your laser keeps wandering off to different wavelengths.
How do you make it stay put?

Laser Locking for 5-Year-Olds

Your laser keeps wandering off to different wavelengths.
How do you make it stay put?

Lock it to an atomic transition! Shine the laser into an atomic sample and detect the transmitted light on a photodiode. Use the resulting signal to build an error signal sensitive to slight deviations from resonance.

Laser Locking for 5-Year-Olds

Your laser keeps wandering off to different wavelengths.
How do you make it stay put?

Lock it to an atomic transition! Shine the laser into an atomic sample and detect the transmitted light on a photodiode. Use the resulting signal to build an error signal sensitive to slight deviations from resonance.

1. Create a signal with two frequency sidebands around the central laser frequency; send this through an atomic sample.
2. Interfere the sidebands with the transmitted beam; the beats contain absorption and dispersion data as an AM signal!
3. Heterodyne with the sidebands to get an error signal.
4. Construct a feedback loop to keep the laser locked!

Table of contents

Table of Contents

Modulating a Signal

- Consider a signal of the form $s(t) = A_c \cos(\omega_0 t + \phi)$.
- Its frequency is constant, so the phase rises as $\omega_0 t + \phi$.
- If we "modulate" the signal by $\omega_0 \mapsto \omega(t) = \omega_0 + m(t)$, this is no longer true: the phase must be realized as an integral.

The modulated signal is therefore written

$$s_m(t) = A_c \cos\left(\int_0^t \omega(\tau)\ \mathrm{d}\tau\right) = \\ = A_c \cos\left(\omega_0 t + \int_0^t m(\tau)\ \mathrm{d}\tau\right). \tag{1.1}$$

Sinusoidal Modulation

Suppose that we modulate sinusoidally: $m(t) = A_m \cos(\Omega t)$. Its integral is $\int_0^T m(\tau)\, \mathrm{d}\tau = \frac{A_m}{\Omega} \sin(\Omega t)$, and so

$$s_m(t) = A_0 \cos\left[\omega_0 t + \frac{\Delta\omega}{\Omega} \sin(\Omega t)\right], \tag{1.2}$$

where $\Delta\omega \equiv A_m$ is called the **peak frequency deviation** and is described by the amplitude—*not* the frequency—of $m(t)$.

This is Chekhov's gun, the loaded pistol gingerly laid down on the table in Act I of a drama. Commit it to mind; it will return.

Bessel Functions

For the modulated signal $s_m(t) = A_0 \cos\left[\omega_0 t + \frac{\Delta\omega}{\Omega}\sin(\Omega t)\right]$, define the **modulation index** $M \equiv \frac{\Delta\omega}{\Omega}$.

Fact of life.

A sinusoidally modulated signal can be expressed in terms of the Bessel functions:

$$s_m(t) = \cos\left[\omega_0 t + \frac{\Delta\omega}{\Omega}\sin(\Omega t)\right] = \sum_{n=-\infty}^{\infty} J_n(M)\cos\left[(\omega_0 + n\Omega)t\right]. \tag{1.3}$$

Bessel Functions: Discussion

We now return to physics; i.e. making ridiculous approximations and claiming that asymptotics make everything obvious.

- In general, $s_m(t)$ has infinite bandwidth, but fortunately the J_n decay quickly: $J_n(M) \approx \frac{M^n}{2^n n!}$, so wide sidebands are small.
- We care about the regime $M \ll 1$ ("narrowband FM"), where $\Delta\omega \ll \Omega$, and the "bandwidth" is about 2Ω. Indeed...
- For $M \ll 1$, the asymptotic $J_n \approx \frac{M^n}{2^n n!}$ permits higher powers M^k to vanish if we chant "to first order" and wave our hands.
- Moreover, we obtain $J_0(M) \approx 1$, $J_{\pm 1}(M) = \pm\frac{M}{2}$, and all of the other Bessel functions are banished to Siberia.

Table of Contents

The Basic Setup

The laser beam provides an electric field

$$\mathbf{E}(t) = \mathbf{E}_0 e^{i\omega_0 t} + c.c.; \tag{2.1}$$

this is promptly modulated as in (1.2). Its frequency now oscillating sinusoidally, the field careens towards the sample:

$$\mathbf{E}_m(t) = \mathbf{E}_0 \exp\left[i\omega_0 t + iM\sin(\Omega t)\right] + c.c., \tag{2.2}$$

where M is the modulation index. In general, as per (1.3),

$$\mathbf{E}_m(t) = \mathbf{E}_0 \sum_{n=-\infty}^{\infty} J_n(M) \exp\left[i(\omega_0 + n\Omega)t\right] + c.c. \tag{2.3}$$

But we are not savages, and for $M \ll 1$ there's a neat trick.

A Smol Approximation

We approximate $E(t)$ to first order in M:

$$E(t) \sim \exp\left[i\omega_0 t + iM\sin(\Omega t)\right] =$$

$$= e^{i\omega_0 t}\exp\left[\frac{M}{2}\left(e^{i\Omega t} - e^{-i\Omega t}\right)\right] \approx$$

$$\approx e^{i\omega_0 t}\left[1 + \frac{M}{2}\left(e^{i\Omega t} - e^{-i\Omega t}\right)\right] \implies$$

$$\boxed{\mathbf{E}(t) = \mathbf{E}_0\left\{e^{i\omega_0 t} + \frac{M}{2}\left[e^{i(\omega_0+\Omega)t} - e^{i(\omega_0-\Omega)t}\right]\right\}.} \qquad (2.4)$$

The Result: Carrier and Sidebands

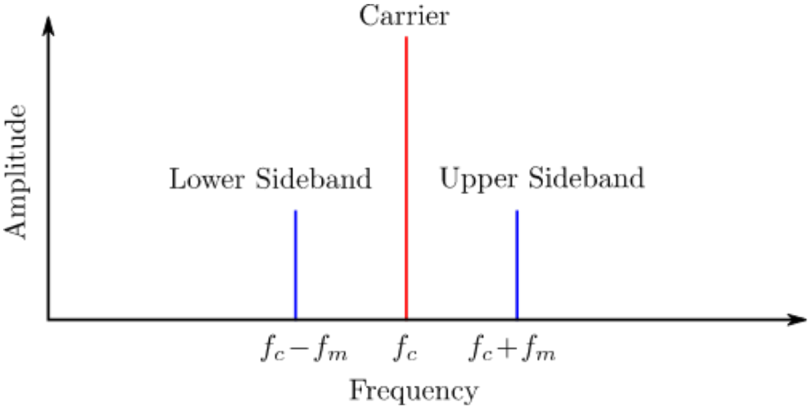


Figure: Small sidebands of amplitude $M/2$ around a carrier frequency. It will be our goal to lock this central frequency to an atomic resonance.

Table of Contents

Transmitted Field

- $\mathbf{E}(t)$ sojourns through an atomic sample with a known resonance.
- The amounts by which its amplitudes and phases are thrown off are functions $\alpha(\omega), \phi(\omega)$ of frequency, but each frequency in the signal is distorted independently.
- A photodiode then detects the intensity of the transmitted field, which scales as $V_T(t) \sim |E_T(t)|^2$. The 3 terms of E_T will interfere, and we will keep terms to first order in M.

Take a deep breath...

Excessive Trigonometry

The transmitted field is

$$E_T(t) \sim \alpha(\omega_0)\, e^{i\omega_0 t}\, e^{i\phi(\omega_0)} + \frac{M}{2}\left[\alpha(\omega_0+\Omega)e^{i(\omega_0+\Omega)t}e^{i\phi(\omega_0+\Omega)} - \alpha(\omega_0-\Omega)e^{i(\omega_0-\Omega)t}e^{i\phi(\omega_0-\Omega)}\right]. \tag{3.1}$$

To first order in M, the photodiode detects

$$V_T(t) \sim \frac{M}{2}\alpha(\omega_0)\left[1 \pm \alpha(\omega_0 \pm \Omega)\left(e^{i\Omega t}e^{\mp i(\phi(\omega_0)-\phi(\omega_0\pm\Omega))} + c.c.\right)\right]. \tag{3.2}$$

Excessive Trigonometry

...However, we can rewrite this by filling seven pages with sines and cosines and my head with existential despair at 4am. The result is:

$$\boxed{V_T(t) \sim \alpha(\omega_0)\left[1 + \frac{M}{2}\left(\Delta\alpha\cos(\Omega t) + \Delta\phi\sin(\Omega t)\right)\right],} \quad (3.3)$$

where we have introduced

$$\Delta\alpha \equiv \alpha(\omega_0 - \Omega) - \alpha(\omega_0 + \Omega); \quad (3.4)$$

$$\Delta\phi \equiv \phi(\omega_0 + \Omega) + \phi(\omega_0 - \Omega) - 2\phi(\omega_0) = $$
$$= 2\left[\overline{\phi}(\omega_0 \pm \Omega) - \phi(\omega_0)\right]. \quad (3.5)$$

What's Going On Here?

- Computing $V_T(t)$ amounts to interfering the sidebands with the transmitted carrier signal. The signal now has an interference envelope (with beat frequency Ω) whose *amplitude* encodes the frequency-dependent absorption and dispersion functions.
- The in-phase component measures the absorption difference between the sidebands, and the quadrature component measures the difference between the average of the sidebands' phase shift and the carrier's phase shift.
- Also, if Ω is small, then the cosine term becomes proportional to the derivative of the absorption, while the sine term becomes proportional to the second derivative of the dispersion.

Table of Contents

Mixing Signals

If only there were a way to extract $\Delta\alpha$ and $\Delta\phi$...

Recall the trigonometric identities

$$\sin(\theta_1 t)\sin(\theta_2 t) = \frac{1}{2}\left[\cos((\theta_1 - \theta_2)t) - \cos((\theta_1 + \theta_2)t)\right]\,; \quad (4.1)$$

$$\sin(\theta_1 t)\cos(\theta_2 t) = \frac{1}{2}\left[\sin((\theta_1 - \theta_2)t) - \sin((\theta_1 + \theta_2)t)\right]\,. \quad (4.2)$$

- If we "mix" two sinusoidal signals by multiplying them, we can then use a low-pass filter to isolate the beat frequency between them.
- Suppose we mixed $V_T(t)$ with an internal-oscillator signal $\sin(\Omega t + \theta)$, where θ is a tunable phase.

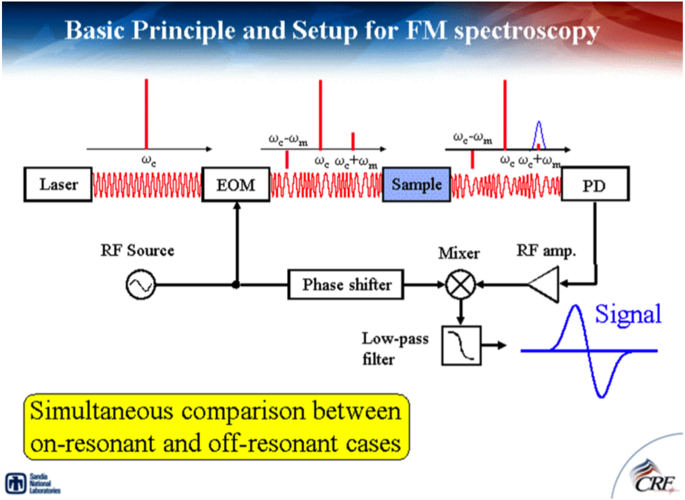
Locking a Laser

Applying a low-pass filter to the mixed signal, we would recover

$$\Delta\alpha \sin(\theta) + \Delta\phi \cos(\theta). \tag{4.3}$$

Since θ is tunable, we can crank it up to π to study absorption, or down to 0 to study dispersion. And indeed, both $\Delta\alpha$ and $\Delta\theta$ have sharp zero crossings when the resonance condition is achieved.

High-Level Diagram



Gaussian Spectral Feature