Entanglement and Entropy

0. Contents

→ Two key tools: tensor products & superposition

$\otimes$ "AND" $+$ "OR"

1. A First Example

Let's take a quantum-mechanical exam.
What strategies do we have?

1. Will it rain tomorrow? $\boxed{A}$ Yes. $\boxed{B}$ No. **2.** Will you wear a raincoat? $\boxed{A}$ Yes. $\boxed{B}$ No.	Options $\boxed{\begin{matrix}1.\ A\\ 2.\ A\end{matrix}}$ $\boxed{\begin{matrix}1.\ B\\ 2.\ B\end{matrix}}$ $\boxed{\begin{matrix}1.\ A \text{ or } B\\ 2.\ A \text{ or } B\end{matrix}}$ $\boxed{\boxed{\begin{matrix}1.\ A\\ 2.\ A\end{matrix}} \text{ or } \boxed{\begin{matrix}1.\ B\\ 2.\ B\end{matrix}}}$

① Classical guess, $|\psi_A\rangle = |\boxed{A}_1\rangle \otimes |\boxed{A}_2\rangle$, or

$$|\psi_B\rangle = |\boxed{B}_1\rangle \otimes |\boxed{B}_2\rangle.$$

② Superposition I, not entangled:

$$|\psi_s\rangle = \frac{1}{\sqrt{2}}\left[|\boxed{A}_1\rangle + |\boxed{B}_1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|\boxed{A}_2\rangle + |\boxed{B}_2\rangle\right].$$

③ Superposition II, entangled:

$$|\psi_e\rangle = \frac{1}{\sqrt{2}}\left[\left(|\boxed{A}_1\rangle \otimes |\boxed{A}_2\rangle\right) + \left(|\boxed{B}_1\rangle \otimes |\boxed{B}_2\rangle\right)\right].$$

<u>Notation:</u>

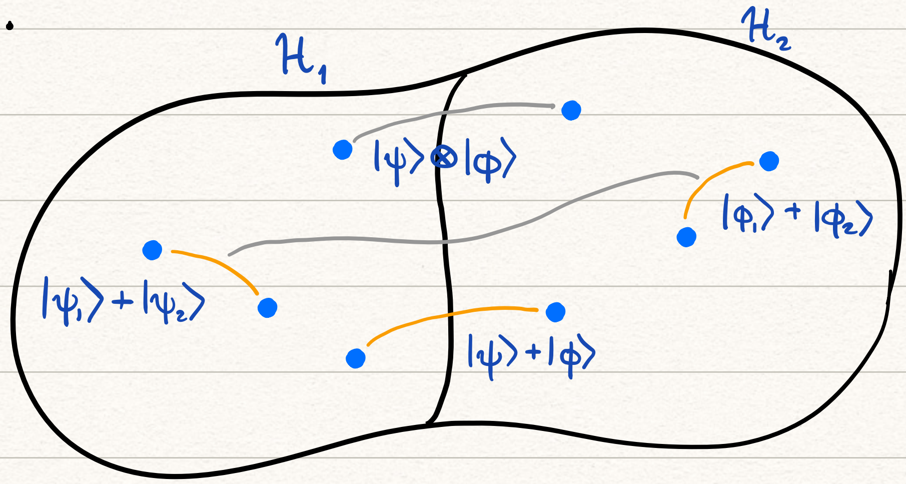
- $|\psi_A\rangle = |\boxed{A}_1\rangle|\boxed{A}_2\rangle = |\boxed{A}_1\boxed{A}_2\rangle$,
- $|\psi_B\rangle = |\boxed{B}_1\rangle|\boxed{B}_2\rangle = |\boxed{B}_1\boxed{B}_2\rangle$;

NOT a product state! →

- $|\psi_s\rangle \propto |\boxed{A}_1 + \boxed{B}_1\rangle |\boxed{A}_2 + \boxed{B}_2\rangle$,
- $|\psi_e\rangle \propto \overbrace{|\boxed{A}_1 \boxed{A}_2\rangle}^{|\psi_A\rangle} + \overbrace{|\boxed{B}_1 \boxed{B}_2\rangle}^{|\psi_B\rangle}$.

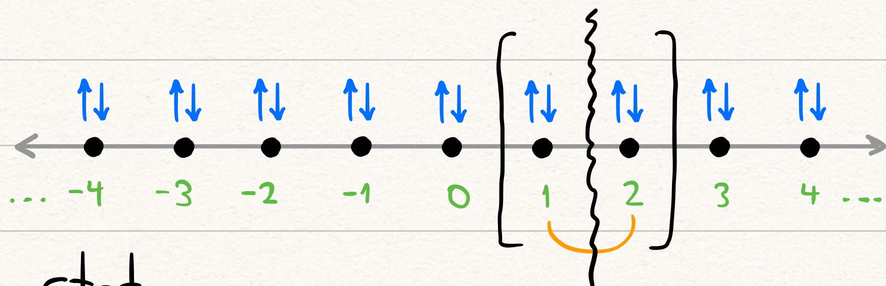
$\boxed{2.}$ Visualizing Entanglement

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.



E.g. Two identical particles.

E.g. Spin chain! [1D lattice]



Q: How do we specify the full state of the system?

A: 3 options:

* Specify ↑ or ↓ on each site
* Build local superpositions of ↑ & ↓
* Superpose states nonlocally, across sites!

E.g. $|\psi_e\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right]$ (EPR state)

Q: How entangled can a state be?

3. DENSITY MATRICES I

[Measuring entanglement: motivation & ideas]

Big idea: nonlocal correlations are stored in the "threads" across tensor factors $\mathcal{H}_i$. (lattice sites i)

Break these threads, then count broken connections. That is, forget one of the tensor factors.

E.g. $|\psi_e\rangle \rightsquigarrow \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_1 (?)_2 - |\downarrow\rangle_1 (?)_2\right] \equiv \mathrm{Red}_1(|\psi_e\rangle)$.

$|\uparrow\rangle_2$ or $|\downarrow\rangle_2$

The resulting object is <u>not</u> a state: it's a probability distribution of states!

Two options, $\uparrow$ or $\downarrow$, for each of the $(?)$'s. Each occurs w/ equal probability $\frac{1}{2}$. Capture all 4 possibilities in a matrix:

first attempt to describe mixed states

$$\rho_1 \overset{?}{=} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

[Reduced density matrix]

<u>WRONG</u>! Do NOT do this.

Better: 1 & 2 are entangled, so forgetting 2 leaves 1 in a classical mixture of $|\uparrow\rangle$ & $|\downarrow\rangle$.

Hence, $\rho_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. ← correct!

$\boxed{4.}$ Density Matrices II

[Measuring entanglement: formalism & calculations]

Density matrices: eigenvalues are probabilities.

Def. A density matrix ρ satisfies 3 properties:

* ρ is Hermitian - ρ has enough eigenvalues
* ρ is positive-definite - eigenvalues are positive
* $\mathrm{Tr}(\rho) = \sum_i \rho_{ii} = 1$. - eigenvalues sum to 1.

Pure states have only one nonzero eigenvalue:

$\rho = \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$. This is a projection:

$\rho = |\psi\rangle\langle\psi|$, and $\rho^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|) =$

$= |\psi\rangle\langle\psi|\psi\rangle\langle\psi| =$

$= |\psi\rangle\langle\psi| = \rho$.

Mixed states contain a distribution $\{\lambda_1, \dots, \lambda_n\}$ of pure states $|\psi_i\rangle$ w/ probabilities λ_i.

How much don't we know? How mixed is ρ? How wide is the probability distribution of λ_i?

Answer: $S_{vN}[\rho] \equiv -\sum_{i=1}^{n} \lambda_i \ln(\lambda_i) = -\mathrm{Tr}[\rho \ln(\rho)]$.

E.g. If $\rho = |\psi\rangle\langle\psi|$ is pure, then $\lambda_i \in \{0, 1\}$, so

$$S_{vN}[\rho] = -\sum_{i=1}^{n} \lambda_i \ln(\lambda_i) = \begin{Bmatrix} 0 \ln(0) = 0 \\ 1 \ln(1) = 0 \end{Bmatrix} = 0.$$

E.g. If $\rho = \frac{1}{n}\mathbb{1} = \begin{pmatrix} \frac{1}{n} & & \\ & \frac{1}{n} & \\ & & \ddots & \\ & & & \frac{1}{n} \end{pmatrix}$, then all $\lambda_i = \frac{1}{n}$, so

$$S_{vN}[\rho] = -\sum_{i=1}^{n} \frac{1}{n} \ln\left(\frac{1}{n}\right) = -\frac{n}{n} \ln\left(\frac{1}{n}\right) = \ln(n).$$

5. Entanglement Entropy

Mixed states arise from (ignorance of) entanglement, so we can use S_{vN} to measure entanglement!

To do so, follow these 3 steps:

① Start with a pure state $\rho = |\psi\rangle\langle\psi|$ in a composite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. It could be entangled, i.e. $|\psi\rangle \neq |\phi_A\rangle \otimes |\phi_B\rangle$.

② Take the partial trace over $\mathcal{H}_B$:
$\rho_A = \mathrm{Tr}_B(\rho) \longleftarrow$ reduced density matrix.

N.B. If $|\psi\rangle = |\phi_A\rangle|\phi_B\rangle$ is a tensor product, then ρ AND ρ_A are pure. Otherwise, ρ_A becomes mixed due to entanglement.

③ The entanglement entropy of ρ on A is the von Neumann entropy of ρ_A:

$$S_A[\rho] \equiv S_{vN}[\rho_A] = -\mathrm{Tr}_A[\rho_A \ln(\rho_A)].$$

$\longrightarrow$ My research: computing S_A in QFT, and using AdS/CFT to make dual statements about quantum gravity.