The Light Cone Frame

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Leonard Susskind once gave the following cheeky relativistic argument. Consider a particle of mass m and momentum $\mathbf{p}$. In units where $c = 1$, its energy is given by $E = \sqrt{\mathbf{p}^2 + m^2}$. If the particle is moving very slowly, then we can approximate its energy as

$$E = \sqrt{\mathbf{p}^2 + m^2} = \sqrt{m^2(1 + \mathbf{p}^2/m^2)} = m\sqrt{1 + \frac{\mathbf{p}^2}{m^2}} \approx m\left(1 + \frac{\mathbf{p}^2}{2m^2}\right) = m + \frac{\mathbf{p}^2}{2m}. \quad (0.1)$$

Here we've applied the binomial approximation to the small quantity $\mathbf{p}^2/m^2$, which is small because we're assuming that the "mc^2" internal energy of the particle is much larger than the energy that its momentum $\mathbf{p}$ gives it. The first term in this expansion is the aforementioned rest energy of the particle, while the second term is the nonrelativistic kinetic energy.

Now, let's turn this approximation on its head. We'll instead boost the particle up by a fixed but extremely large amount in the z direction, so that in the formula

$$E = \sqrt{p_z^2 + p_x^2 + p_y^2 + m^2}, \quad (0.2)$$

p_z is by far the largest contribution; everything else pales in comparison. We carry out a binomial expansion as before: the result is

$$E = \sqrt{p_z^2\left(1 + \frac{p_x^2 + p_y^2 + m^2}{p_z^2}\right)} \approx p_z\left(1 + \frac{p_x^2 + p_y^2 + m^2}{2p_z^2}\right) = p_z + \frac{p_x^2 + p_y^2}{2p_z} + \frac{m^2}{2p_z}. \quad (0.3)$$

How should we interpret this? Well, p_z is very large, so is $E = p_z + \cdots$, which makes sense since the particle moves very fast. But by the same token, we've boosted the particle up so fast that we "washed out" any dynamics happening in the z-direction. In other words, p_z is constant, while p_x and p_y remain dynamical. And by conventional wisdom, i.e. since only energy differences matter in physics, we're allowed to drop such constants from the total energy. The remaining energy (allowing m to be nonconstant for now) is

$$E = \frac{p_x^2 + p_y^2}{2p_z} + \frac{m^2}{2p_z} \equiv \frac{p^2}{2p_z} + \frac{m^2}{2p_z}. \quad (0.4)$$

Here we've introduced $p^2 \equiv p_x^2 + p_y^2$ as a label for the magnitude of the transverse (xy) momentum. Since $E \sim 1/p_z$ and p_z is large, we see that energy differences are small. This makes sense too: most of the particle's energy has been thrown into moving it in the z direction, leaving only small variations in the xy plane. We can understand these small energies in two ways: relativistically and nonrelativistically.

First, let us note that there is a deep association between energy (E) and time evolution (i.e. time derivatives $\frac{\partial}{\partial t}$). This association can be made precise in both classical and quantum mechanics, but for now let's just imagine that $E \sim \frac{\partial(\text{physics})}{\partial t}$. If E, and hence $\frac{\partial(\text{physics})}{\partial t}$, is small, then the deliberately vague quantity "physics" is slow-moving. But this is nothing other than relativistic time dilation! By accelerating the particle in the z direction, we move it into a highly boosted reference frame where clocks run slow, and the motion of the particle in the xy plane appears slowed down. The difference between this and special relativity is that the mechanism responsible for time dilation here is not the Lorentz transformations, but rather just the relative scarcity of energy available for moving in the xy plane.

But we can be even more simpleminded and more insidious. The expression (0.4) resembles (0.1); indeed, $p^2/2p_z$ is like $\mathbf{p}^2/2m$, with $\mathbf{p}^2 = p_x^2 + p_y^2 + p_z^2$ replaced by $p^2 = p_x^2 + p_y^2$, and with m replaced by p_z. The interpretation seems to be that the physics in the transverse plane is governed by a *nonrelativistic* energy formula, with p_z playing the role of a mass! And even this makes sense: motion in the xy plane is slow enough, compared to p_z, that it can be treated nonrelativistically; p_z acts as a mass in the sense that it provides a sort of inertia, robbing the xy plane of energy for motion and slowing things down in that plane.

What just happened here is a toy example of holography, according to Susskind. We took a relativistic system in 3D and rewrote it as a nonrelativistic system in 2D. The momentum p_z representing the third dimension was "encoded" in the xy plane, as if in a holographic screen, by the effective mass of particles moving in the xy projection of space. This is, of course, not really how holographic duality (sometimes called AdS/CFT correspondence) works; nevertheless, there are some striking resemblances. As for the second term $m^2/2p_z$, we could have dropped it from (0.4) under the assumption that m is constant. It was retained only for the purpose of pointing out that—to the extent that it can be treated as a relativistic rest energy—it scales with mass *squared*, instead of with mass (as might be expected from $E = mc^2$). The explanation for this scaling lies in string theory; it is related to what particle physicists in the 60s called *Regge trajectories*. But that is a story for another time.