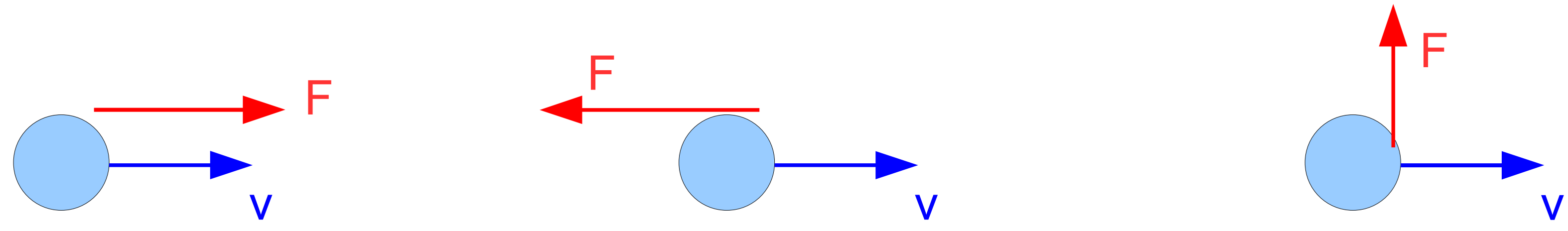
ENERGY & MOMENTUM

Time Evolution of Speed

- Newton's 2nd Law:
  - Describes how force affects velocity of the CM
- How do forces affect speed of the CM?
  - Depends on relative direction of force and velocity:



Speed increases   Speed decreases   Speed is unaffected

- Effect of force on speed is determined by $\vec{F} \cdot \vec{v}$
  - This quantity is called "power" – measured in Watts (W)

Momentum and Kinetic Energy

- Take a closer look at force and power:

"Momentum" (**p**) of the system – measured in kg (m / sec)

$$\vec{F} = m\,\vec{a} = m\,\frac{d\,\vec{v}}{dt} = \frac{d}{dt}\left(m\,\vec{v}\right)$$

$$\vec{F}\cdot\vec{v} = \left(m\,\vec{a}\right)\cdot\vec{v} = \left(m\,\frac{d\,\vec{v}}{dt}\right)\cdot\vec{v} = \frac{d}{dt}\left(\frac{1}{2}\,m\,\left(\vec{v}\cdot\vec{v}\right)\right) = \frac{d}{dt}\left(\frac{1}{2}\,m\,v^2\right)$$

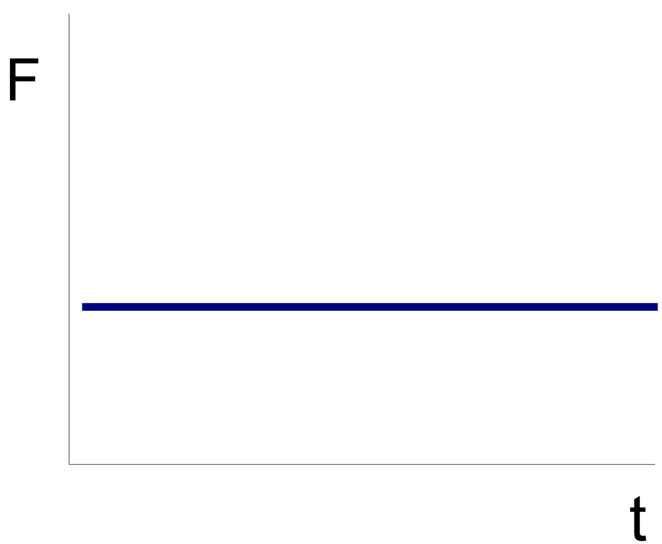
"Kinetic Energy" (KE) of the system – measured in Joules (J)

- Alternative view of Newton's 2nd Law:
  - Force affects the momentum vector of a system
  - Power affects the KE of a system

Forces – “Time of Action”

Forces in physics cover a wide range of timescales:

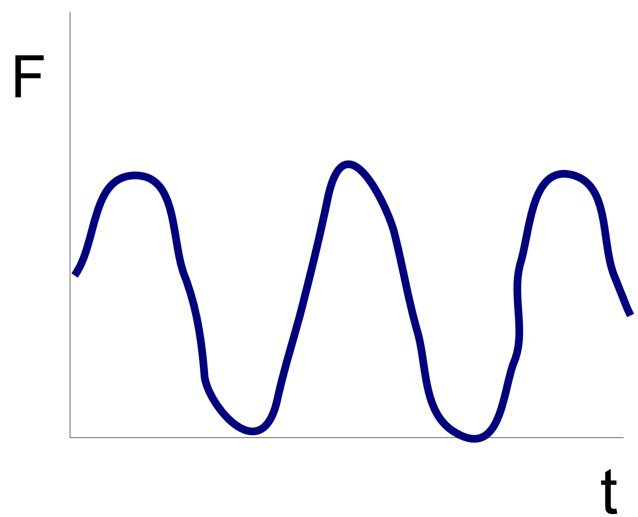
“Steady Forces”



Examples:

- Earth-Moon gravity
- Lift force on a plane

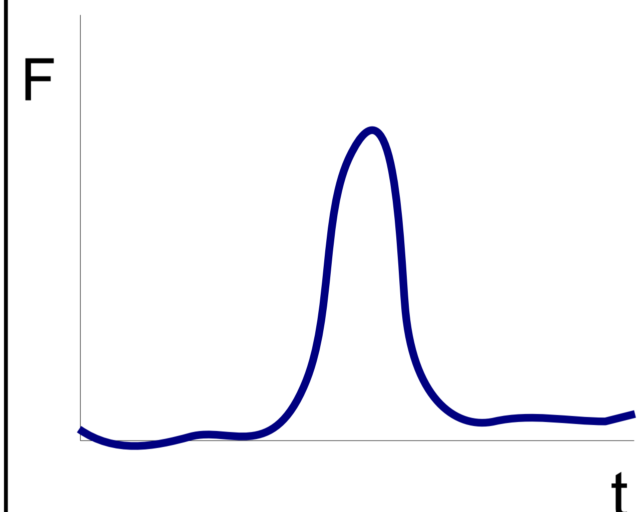
“Periodic Forces”



Examples:

- Sound waves
- Elastic forces (springs)

“Impact Forces”



Examples:

- Collisions
- Explosions

Impulse and Work

- Apply Newton's 2nd Law over a time interval dt:

$$\vec{F}\,dt = d\left(m\,\vec{v}\right) \longrightarrow \int \vec{F}\,dt = m \int d\,\vec{v} \longrightarrow \boxed{\int \vec{F}\,dt = m\left(\vec{v}_f - \vec{v}_i\right)}$$

"Impulse" delivered to the system

- Power over a time interval dt:

$$\vec{F}\cdot\vec{v}\,dt = d\left(\frac{1}{2}\,m\,v^2\right) \longrightarrow \vec{F}\cdot d\,\vec{r} = d\left(\frac{1}{2}\,m\,v^2\right) \longrightarrow \boxed{\int \vec{F}\cdot d\,\vec{r} = \frac{1}{2}\,m\left(v_f^{\,2} - v_i^{\,2}\right)}$$

"Work" delivered to the system

- Terminology:
  - Impulse is the change in momentum of a system
  - Work is the change in KE of a system
  - Given a motion path, either can be calculated

Work / Impulse Example

- A mass m follows the motion path: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} At \\ Bt^2 \end{pmatrix}$
  - Calculate the impulse and work delivered between $t=t_1$ and $t=t_2$
  - Do it both ways:
  - 1) using the velocity vector
  - 2) by calculating the force vector and integrating

- Think of some physical examples that would produce this motion path

- Set a tomato on a golf tee and shoot it with a bullet
  - Tomato explodes, but doesn't move far from tee
  - Explain in terms of work and impulse

Momentum/KE of Physical Systems

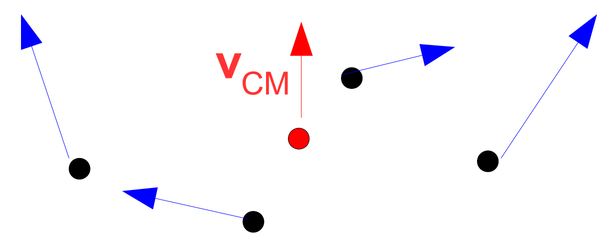
- Physical systems can consist of <u>many</u> particles
  - Each with their own mass and velocity
  - Calculate momentum/KE of the system from its <u>particles</u>:

$$\vec{p}_{system} = \sum_{i=1}^{N} m_i \, \vec{v}_i = \sum_{i=1}^{N} m_i \left(\vec{v}_{CM} + \vec{v}_i \,' \right)$$

$$\vec{p}_{system} = \sum_{i=1}^{N} m_i \, \vec{v}_{CM} + \sum_{i=1}^{N} m_i \, \vec{v}_i \,'$$

0 (by definition of the CM)

$$\boxed{\vec{p}_{system} = M_{system} \, \vec{v}_{CM}}$$



$$KE_{system} = \sum_{i=1}^{N} \frac{1}{2} m_i \, v_i^{\,2} = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\vec{v}_{CM} + \vec{v}_i \,' \right) \cdot \left(\vec{v}_{CM} + \vec{v}_i \,' \right)$$

$$KE_{system} = \sum_{i=1}^{N} \frac{1}{2} m_i \, v_{CM}^{\,2} + \sum_{i=1}^{N} \frac{1}{2} m_i \left(v\,'_i \right)^2 + \sum_{i=1}^{N} m_i \left(\vec{v}_{CM} \cdot \vec{v}_i \,' \right)$$

0 (by definition of the CM)

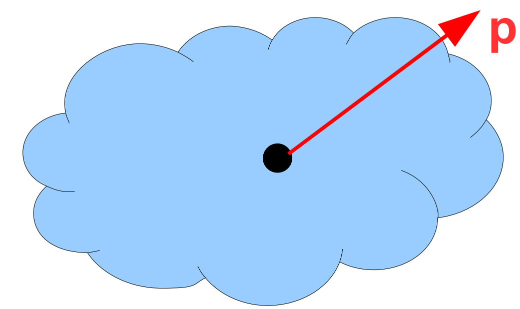
$$\boxed{KE_{system} = \frac{1}{2} M_{system} \, v_{CM}^{\,2} + \sum_{i=1}^{N} \frac{1}{2} m_i \left(v\,'_i \right)^2}$$

KE in form of rotation, thermal motion, explosion, etc.

Momentum/KE of Mass Distributions

- "Velocity" at each point of a mass distribution:
  - Not well-defined quantity – impossible to "track" moving mass

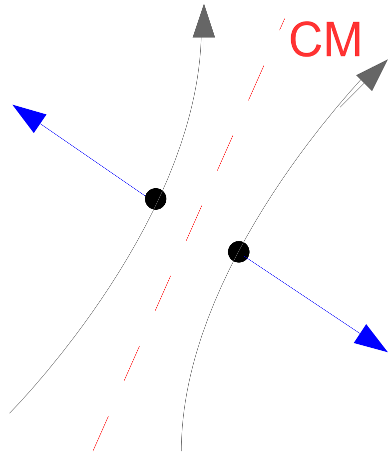
- Total momentum of a mass distribution:
  - Can be defined: $\vec{p}_{total} = M_{total}\,\vec{v}_{CM}$



- Total KE of a mass distribution:
  - Tougher to understand: $KE_{total} = \frac{1}{2} M_{total}\, v_{CM}^{2} + KE_{internal}$
  - $KE_{internal}$ can be defined – not necessarily by Newton's Laws

- Examples
  - Elastic media, rigid body rotation – define **v** using assumptions
  - Quantum mechanics – uses only momentum and KE (not **v**)

Consequences of Newton's 3rd Law



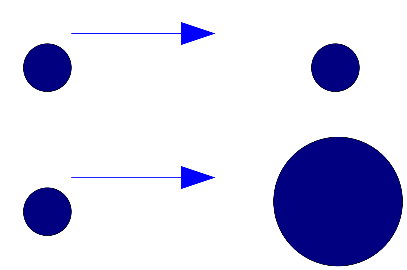
$$\vec{F}_{12} = -\vec{F}_{21} \longrightarrow \frac{d\vec{p}_1}{dt} = \frac{-d\vec{p}_2}{dt}$$

$$\boxed{\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0}$$

$$\boxed{\frac{d}{dt}\left(M_{total}\,\vec{v}_{CM}\right) = 0}$$

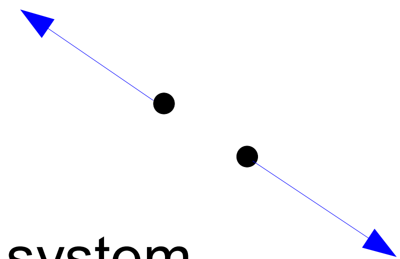
- <u>Conservation of Momentum:</u>
  - External force changes system's momentum in some <u>direction</u>
  - Reaction force exerted <u>by</u> that system:
  - Changes <u>another</u> system's momentum in opposite direction

- <u>Example</u>: Collisions – reflection vs. absorption
  - Reflecting particle delivers 2x as much impulse
  - Re-work this problem in the "CM frame"



Forces and Potential Energy

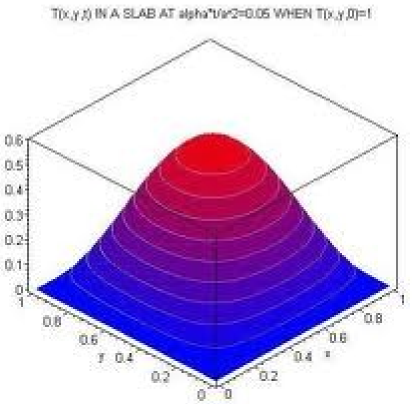
- Consider 2 particles with action/reaction pair of forces
  - Consider the system made up of both particles:
  - Net external force is zero → $\mathbf{v}_{CM}$ is constant
  - Internal forces can not change total momentum of system
  - But internal forces can change KE in this term: $\sum_{i=1}^{N} \frac{1}{2} m_i \left(v'_i\right)^2$



- Results of experiments on forces:
  - Forces are usually determined by relative location of particles
  - Arrangement of particles → forces → change in KE
- "Potential Energy" (PE)
  - The amount of KE to be gained/lost when forces act
  - PE is a function of the relative location of particles

Mathematical Description of PE: “Fields”

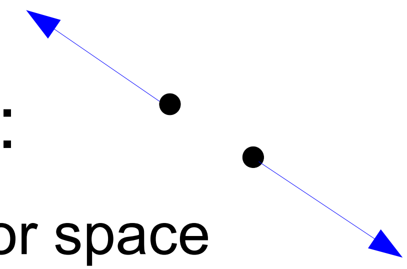
- Mathematical definition: “field”
  - Function which maps every point in a vector space to a value
  - Value could be a single number (“scalar”), vector, tensor, …



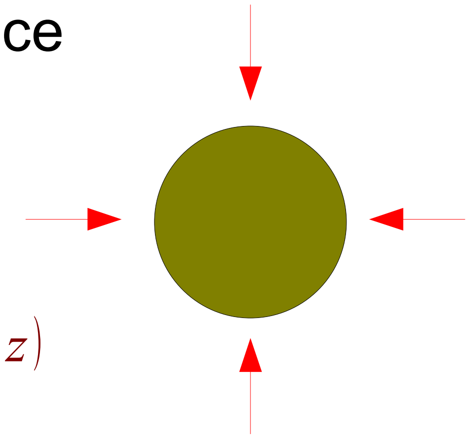
- Example: Temperature in a room
  - Scalar field – one number for each position vector
  - In 2 different reference frames:
  - Functional form of $T(x,y,z)$ will be different...
  - ...but temperature at center of room must be the same

- What is the derivative of a field like $T(x,y,z)$?
  - Must be expressed separately in the (x,y,z) directions
  - “Gradient” $\rightarrow$ $\nabla T(x,y,z) = \left(\frac{\partial T}{\partial x}\right)\hat{i} + \left(\frac{\partial T}{\partial y}\right)\hat{j} + \left(\frac{\partial T}{\partial z}\right)\hat{k}$

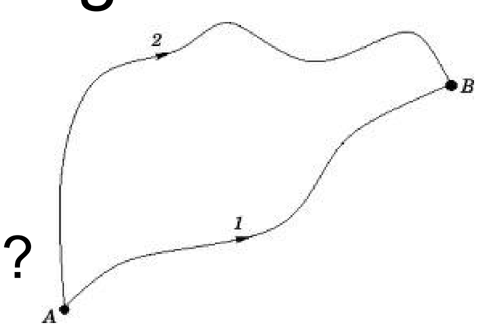
PE and "Force Fields"



- For 2 particles exerting forces on each other:
  - Their relative position ($\vec{r}_1 - \vec{r}_2$) makes up a vector space
  - The force $\mathbf{F}_{12}$ is a vector field in that vector space

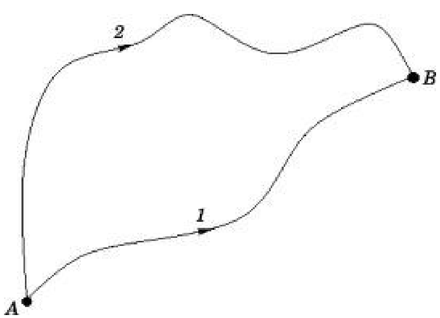


- Example: Gravitational field of a planet
  - All of space is filled with a vector field $\vec{g}\,(x\,,\,y\,,\,z)$
  - Pointing toward the origin (center of planet)
  - Actual force is exerted only when another particle enters field
- Forces push particles on motion paths → doing work
  - In other words: Forces convert PE into KE
  - Consider work done along 2 different paths:
  - Does work depend on the path taken from A to B?



Conservative Force Fields

- For "conservative" force fields:
  - Work done by force is independent of path
  - So PE only depends on the arrangement of the particles
  - Therefore PE is a scalar field → often written U(x,y,z)



$$\left.\begin{aligned} \Delta U_1 &= -Work_1 = -\int_A^B \vec{F}\cdot d\vec{r}_1 \\ \Delta U_2 &= -Work_2 = -\int_A^B \vec{F}\cdot d\vec{r}_2 \end{aligned}\right\} \Delta U_1 = \Delta U_2$$

- Conservation of Energy: (with KE denoted as "K")

$$Work = \Delta K \longrightarrow \Delta U = -\Delta K \longrightarrow \Delta(K + U) = 0$$

$$\boxed{K + U = E_{total} = Constant}$$

Note: Conservation of Energy can not determine what the value of E_{total} is – just that it is constant

U(x,y,z) and U(x,y,z) + U_0 describe the same force field

Force and PE: Mathematical Relation

- Consider work due to a small displacement:

$$dW = -dU = -\left[\left(\frac{\partial U}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y}\right)dy + \left(\frac{\partial U}{\partial z}\right)dz\right]$$

$$dW = \vec{F}\cdot d\vec{r} = \left[F_x\, dx + F_y\, dy + F_z\, dz\right]$$

- By inspection:

$$\vec{F} = -\left[\left(\frac{\partial U}{\partial x}\right)\hat{i} + \left(\frac{\partial U}{\partial y}\right)\hat{j} + \left(\frac{\partial U}{\partial z}\right)\hat{k}\right]$$

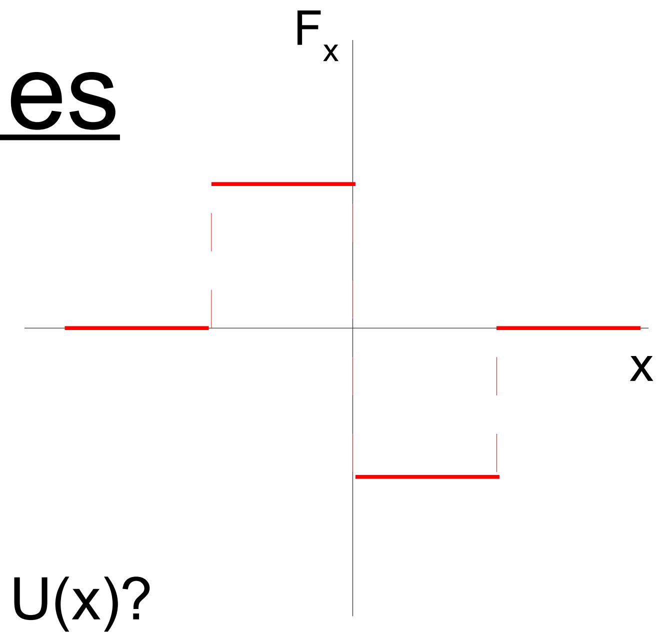
$$\boxed{\vec{F} = -\vec{\nabla} U}$$

Valid for all conservative force fields

$$\boxed{\Delta U_{AB} = -\int_A^B \vec{F}\cdot d\vec{r}}$$

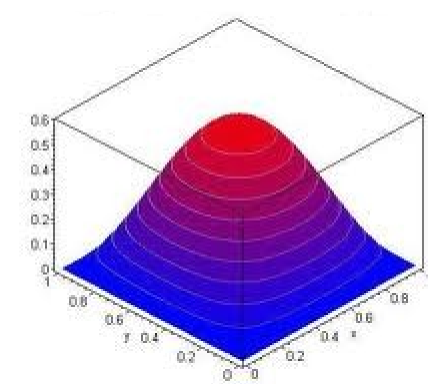
- Force points in the direction of greatest decrease of U

Force Field Examples



- 1-Dimensional: $F_x(x) = \begin{cases} F_0 & -L < x < 0 \\ -F_0 & 0 < x < L \\ 0 & everywhere\ else \end{cases}$
  - Calculate U(x) and sketch graph
  - Can $F_x(x)$ be discontinuous? What about U(x)?

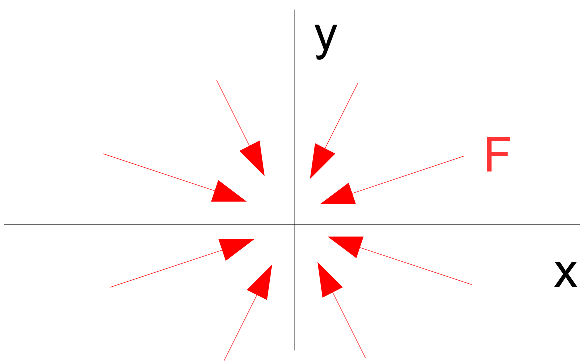
- 2-Dimensional: $U(x, y) = U_0 e^{\frac{-(x^2+y^2)}{a^2}}$
  - Calculate $\vec{F}(x, y)$
  - Calculate work done by force in moving from (0, a) to (2a, 0):
  - Both ways: 1) Path integral and 2) using U(x,y)

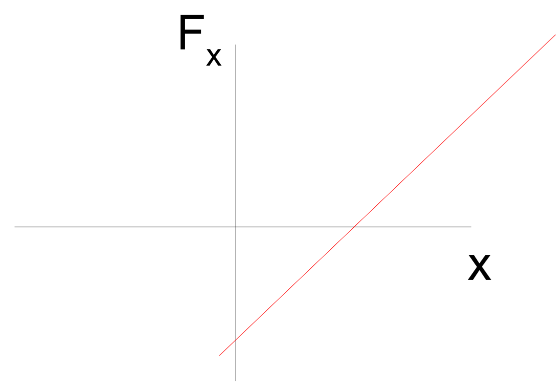


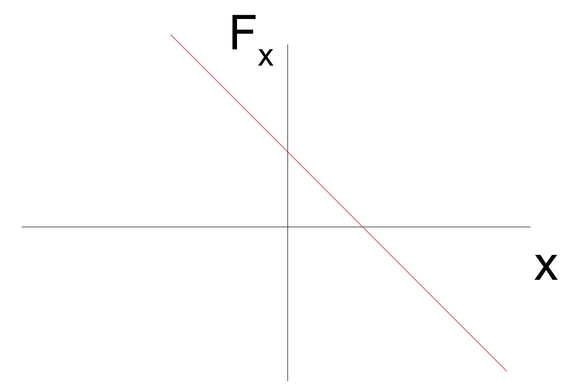
Equilibrium and Stability

- Equilibrium – $F_{net}=0$, or $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = 0$

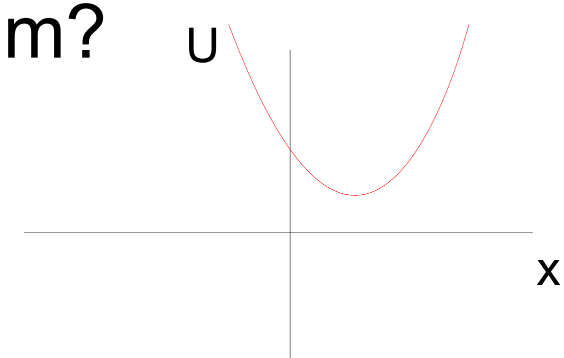
- Stability – force field pushes <u>toward</u> stable equilibrium







- Which of the above is a stable equilibrium?

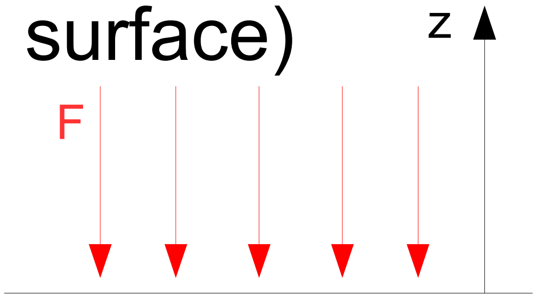


$$\boxed{\left.\frac{\partial F_x}{\partial x}\right|_{equil} \le 0} \quad \longrightarrow \quad \boxed{\left.\frac{\partial^2 U}{\partial x^2}\right|_{equil} \ge 0}$$

  – Possible to be <u>stable</u> in x-direction but <u>unstable</u> in y-direction!

Common Forms of PE

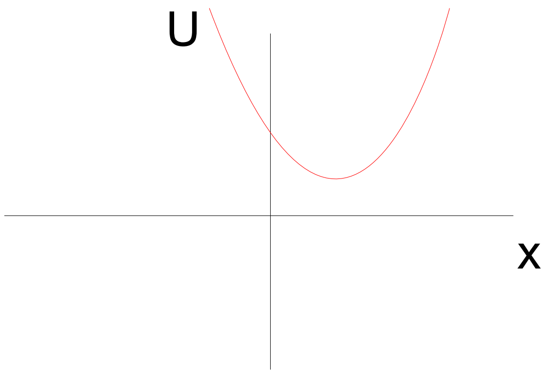
- Uniform force field (e.g. gravity near Earth surface)
  - For general F_0: $\Delta U = F_0 \Delta z$
  - For gravity near Earth surface: $\boxed{\Delta U = mg \Delta z}$



- Elastic PE: springs and elastic media $\longrightarrow$ $F_x = -k\left(x - x_0\right)$

$$\Delta U = -\int_{x_0}^{x_f} F_x \, dx = \frac{1}{2} k \left(x_f^{\,2} - x_0^{\,2}\right) - k\, x_0 \left(x_f - x_0\right)$$

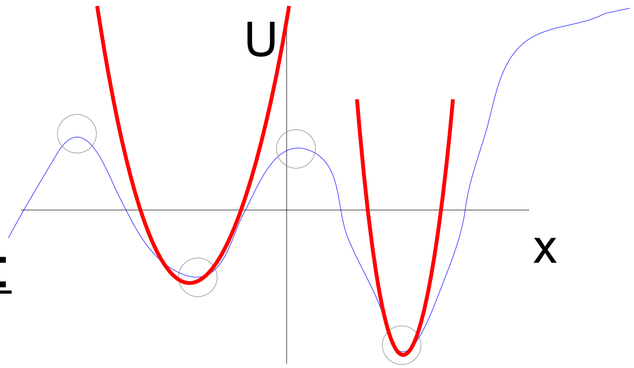
$$\boxed{\Delta U = \frac{1}{2} k \left(x_f - x_0\right)^2}$$



  - Elastic systems typically have at least one stable equilibrium

"Effective" Spring Constant

- Consider an arbitrary U(x):
  - Can contain many equilibrium points
  - Taylor expansion of U(x) near equilibrium:



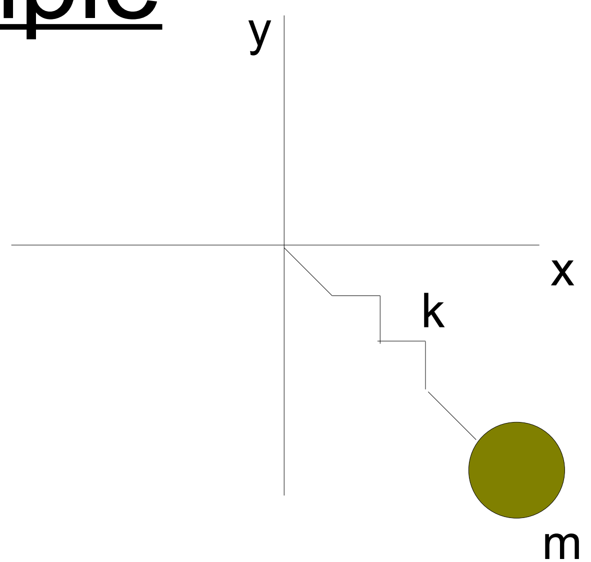
$$U(x) \approx U(x_0) + \left.\frac{\partial U}{\partial x}\right|_{x_0}(x - x_0) + \frac{1}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x_0}(x - x_0)^2 + \ldots$$

Physically meaningless

0

- Near any stable equilibrium
  - Can approximate U(x) using a parabola
  - Which is exactly the PE function for elastic forces!
  - "Effective Spring Constant": $k_{eff} = \left.\frac{\partial^2 U}{\partial x^2}\right|_{x_0}$
- Example: Calculate k_{eff}: $U(x) = \frac{C_1}{x} + C_2 x^3$

Force and PE Example



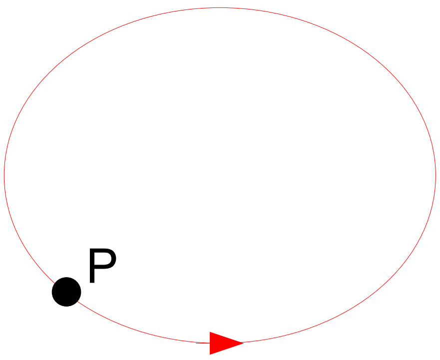
- The "spring pendulum":
  - Unstretched spring has length L
  - In a uniform gravitational field g
- Calculate U(x,y)
  - Use derivatives to find equilibrium point(s)
  - Check stability in both x and y directions
- Are there any locations (x,y) such that:
  - Net force points in the negative x-direction?
  - Net force is perpendicular to position vector?

Non-Conservative Forces

- Not all forces can be assigned a scalar field U(x,y,z)
  - Work is path-dependent for some forces

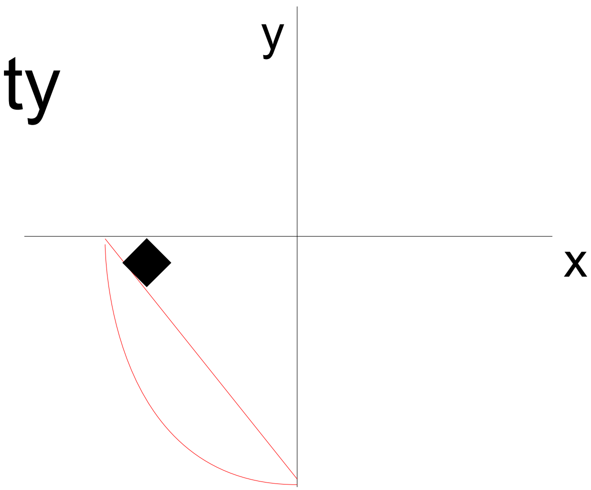
- Example: forces which depend on velocity
  - Kinetic friction, air resistance, viscosity, etc.

- Consider work done on a closed path
  - i.e. a path that starts and ends at same point



  - For a conservative force: $\oint \vec{F} \cdot d\vec{r} = 0$
  - For a non-conservative force: $\oint \vec{F} \cdot d\vec{r} \neq 0$

Non-Conservative Force Field Example



- Block slides down incline due to gravity
  - With friction coefficient μ_k
  - Slide: from (-1.0m, 0) to (0, -1.0m)
  - Calculate work done by friction:
  - 1) along linear path
  - 2) along circular path centered at origin – warning: not easy!

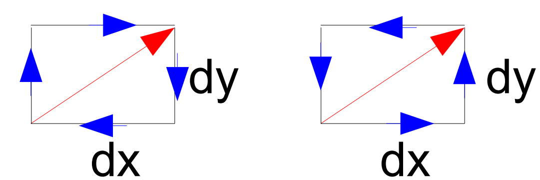
- In general, work is a "functional"
  - Mathematical object → maps one or more functions to a number
  - Functionals often take the form of an integral:

$$F\left[x(t), y(t)\right] = \int g(x, y)\, dt \qquad Work\left[x(t), y(t)\right] = \int \left(F_x \frac{dx}{dt} + F_y \frac{dy}{dt} \right) dt$$

Force Fields – Conservative or Not?

- Force field → can write $\vec{F}(x, y, z)$
  - Non-conservative → can't write $U(x, y, z)$

- What conditions make a force field conservative?
  - Consider a small piece of a path $d\vec{r}$:
  - Conservative if $\oint \vec{F} \cdot d\vec{r} = 0$
  - Must be true if path is traversed CW or CCW



CW: $F_y(x, y)\,dy + F_x(x, y+dy)\,dx - F_y(x+dx, y+dy)\,dy - F_x(x+dx, y)\,dx = 0$

CCW: $F_x(x, y)\,dx + F_y(x+dx, y)\,dy - F_x(x+dx, y+dy)\,dx - F_y(x, y+dy)\,dy = 0$

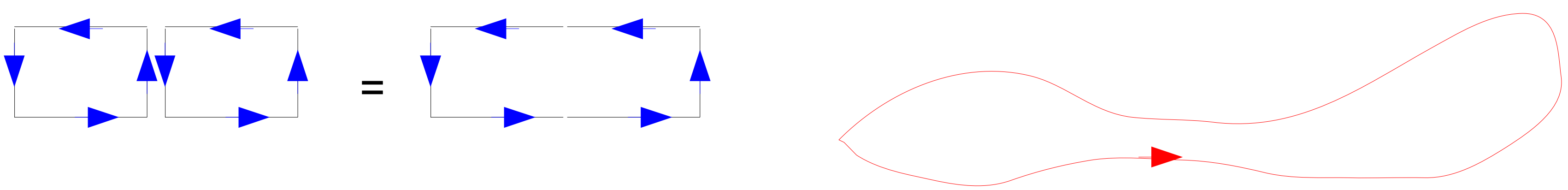
Taylor series to 1st order (example term): $F_x(x+dx, y+dy) = F_x(x, y) + \frac{\partial F_x}{\partial x}dx + \frac{\partial F_x}{\partial y}dy$

Plug in and subtract (CW – CCW): $\boxed{\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0}$

Implies that work is independent of path for this dx and dy

Force Fields – Conservative or Not?

- Consider work done along 2 neighboring closed paths:



- Can add work from <u>many</u> paths to construct any path
  - If $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$ everywhere → can write U(x,y)
- In 3 dimensions: $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0 \quad \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} = 0 \quad \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0$

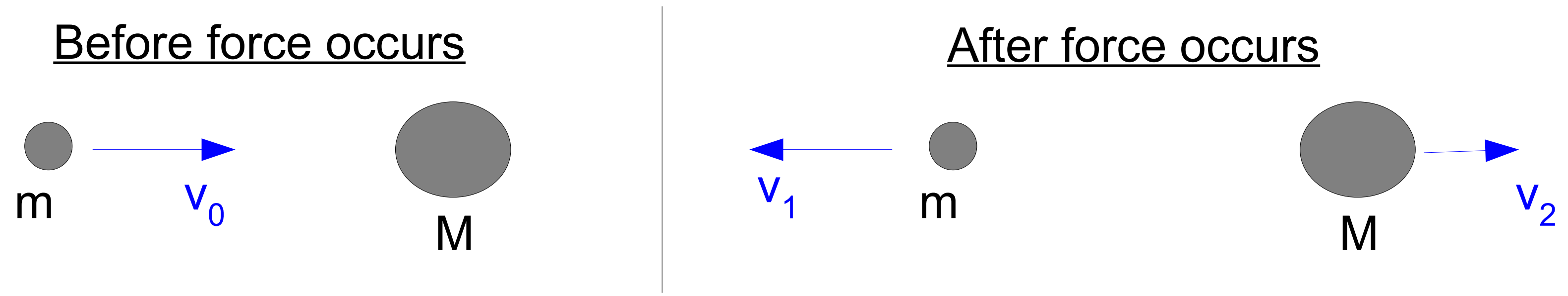
"Curl" of a vector field: $\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{i} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}$

  - Force field is conservative if $\boxed{\vec{\nabla} \times \vec{F} = 0}$

Collisions / Explosions

- Import application of conservation laws
  - Can compare momentum and energy before/after an "event"

- Example: "Impact" force with very small range



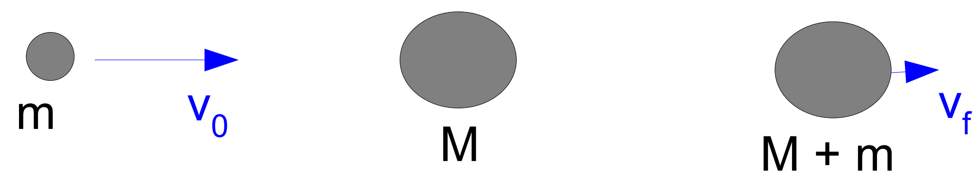
- Using conservation of momentum/energy:
  - Can determine relationship(s) between velocities before/after
  - Without knowing details of force!
  - "Elastic" collisions → No KE converted to/from PE → $KE_f = KE_i$
  - "Inelastic" collisions → Must account for PE (in many forms)

Collision/Explosion Examples

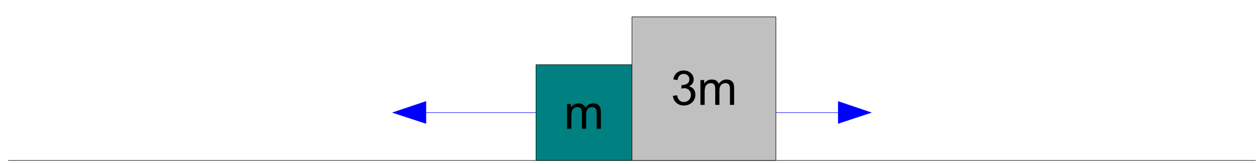
- Perfectly inelastic collision:

Before collision

After collision



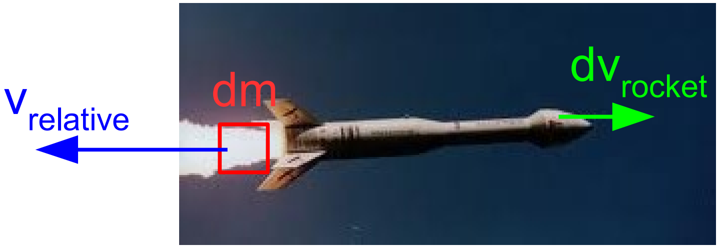
  - Calculate velocity of final particle after collision
  - Could this collision be elastic or inelastic? (Hint: CM frame)
  - If inelastic, how does $E_{internal,f}$ compare to $E_{internal,i}$?
  - Compare this with electron/proton colliding to form H atom



- Explosion:
  - Firecracker explodes between two blocks, releasing energy E_0
  - Calculate speed of each block immediately after explosion
  - Each block has friction coefficient $\mu_k \rightarrow$ find where each stops

Rocket Equation

- Momentum conservation can be applied to propulsion:
  - Vehicle pushes gas, liquid, or plasma in one direction...
  - ...and experiences a force in the opposite direction



$$\left(m_{rocket} + m_{remaining\ fuel}\right) d\,\vec{v}_{rocket} = -(dm)\,\vec{v}_{relative} + \vec{F}_{external}\,dt$$

- Example: Rocket which starts at rest and ejects fuel...
  - With constant exhaust speed $v_{exhaust}$ at constant rate R (kg/s)
  - Calculate v(t) of the rocket

$$\left.\begin{aligned} \left(M_{total}(t)\right)\frac{d\,\vec{v}_{rocket}}{dt} &= -\frac{dm}{dt}\,\vec{v}_{exhaust} \\ M_{total}(t) &= m_{rocket} + m_{fuel,0} - R\,t \end{aligned}\right\} \quad dv_{rocket,x} = v_{exhaust}\frac{R\,dt}{m_{rocket} + m_{fuel,0} - Rt}$$