KINEMATICS

Mathematical Description of Time

- Vector spaces have an extra "invisible coordinate"
  - Its coordinate axis has universal positive direction (unlike xyz)
  - Every vector in space must add time to its list of coordinates
  - 3-D points – not good enough, must use 4-D "events" (ct,x,y,z)
  - c (units: length/time) → all coordinates must have same units

- Geometry must be generalized to include time
  - Path in space → set of points (x,y,z) which meet conditions
  - Motion Path in "spacetime" → events (ct,x,y,z) meet conditions

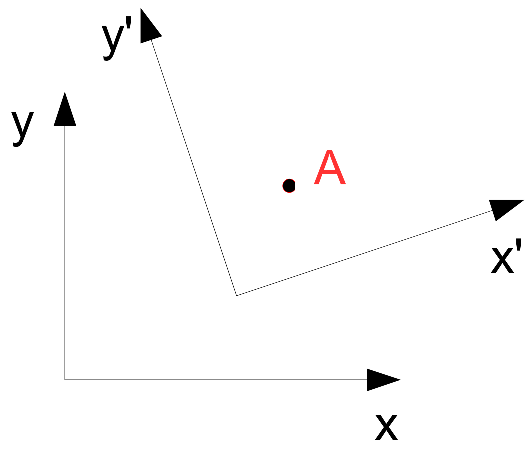
- Example:

$$(0,0,0,0)$$
$$(1,1,1,1)$$
$$(2,2,2,2)$$

Events on a motion path which crosses the origin at time t=0, then moves in the spatial direction (1,1,1) at a particular speed

Newtonian View of Time

- Time t – completely "separable" from other coordinates
  - Time coordinate of events is independent of reference frame



Event A has coordinates (ct, x, y) in one reference frame and (ct', x', y') in another

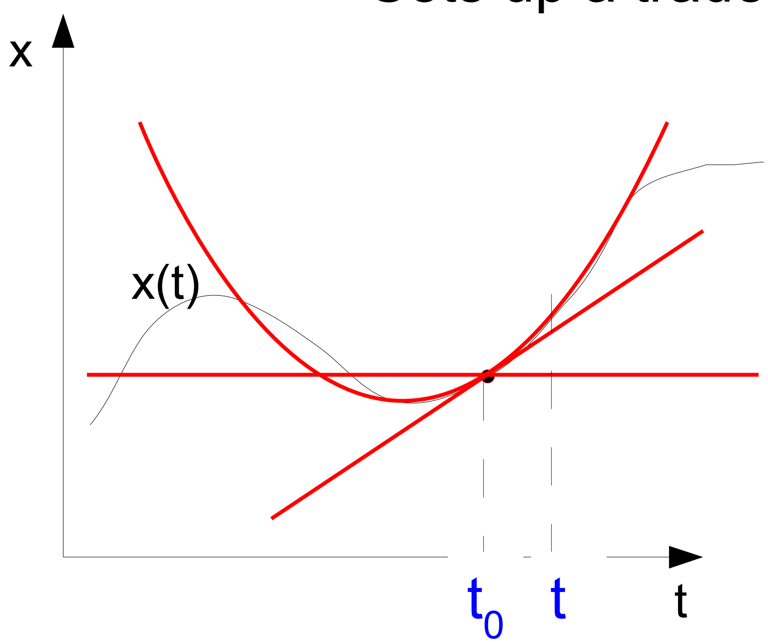
Newtonian view: $t' = t$ for all events

  - Vectors have components only in spatial directions: $\vec{A}(t) = \begin{pmatrix} A_x(t) \\ A_y(t) \\ A_z(t) \end{pmatrix}$
  - t is called a "parameter", not a coordinate → don't need c

- Einstein's Relativistic view of time:
  - Time coordinate ct depends on reference frame (just like x,y,z)
  - Vectors have a component in the "time direction" (4-vectors)

Describing Motion Paths Mathematically

- Time interval of interest is typically not infinite
  - i.e. need to describe $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ in the vicinity of some time t_0
- Useful tool: "Taylor Series"
  - Organizes complicated function x(t) into simpler parts (near t_0)
  - Sets up a trade-off between simplicity and precision



Example: Driver of a car randomly presses gas and brake pedal. At time t_0 the car is at position x_0. Where is the car:

At time t_0? (Duh) $x = x_0$ (1 number, valid at 1 instant)

0.1 seconds later? $x \approx x_0 + v_0 \left(t - t_0\right)$

1.0 seconds later? $x \approx x_0 + v_0 \left(t - t_0\right) + \frac{1}{2} a_0 \left(t - t_0\right)^2$

Motion Paths and Taylor Series

- Taylor Series: Coefficient of n^{th} term...
  - ...is related to n^{th} derivative of function x(t), evaluated at t_0

$$x(t) = x_0 + \left.\frac{dx}{dt}\right|_{t_0}(t-t_0) + \left(\frac{1}{2!}\right)\left.\frac{d^2x}{dt^2}\right|_{t_0}(t-t_0)^2 + \left(\frac{1}{3!}\right)\left.\frac{d^3x}{dt^3}\right|_{t_0}(t-t_0)^3 + \dots$$

- Similarly, for a position vector **r**:

$$\vec{r}(t) = \vec{r}_0 + \left.\frac{d\vec{r}}{dt}\right|_{t_0}(t-t_0) + \left(\frac{1}{2!}\right)\left.\frac{d^2\vec{r}}{dt^2}\right|_{t_0}(t-t_0)^2 + \left(\frac{1}{3!}\right)\left.\frac{d^3\vec{r}}{dt^3}\right|_{t_0}(t-t_0)^3 + \dots$$

position at t_0

Instantaneous velocity at t_0

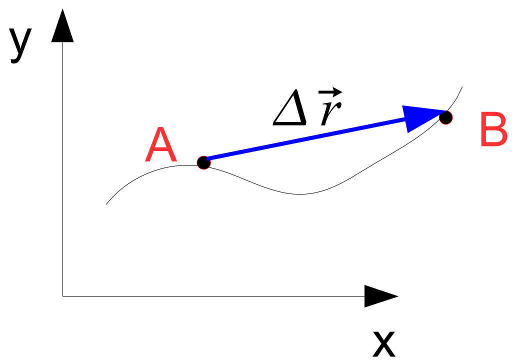
Instantaneous acceleration at t_0

Instantaneous "jerk" at t_0

- Note: t_0 is not always the same as t=0

Velocity – Conventions and Terminology

- Any 2 events (points and instants) on a motion path:
  - Have a displacement vector and a time interval between them



Event A: $\vec{r}_A = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$

time = t_A

Event B: $\vec{r}_B = \begin{pmatrix} x_A + \Delta x \\ y_A + \Delta y \end{pmatrix}$

time = $t_A + \Delta t$

- Average velocity (between A and B): $\vec{v}_{AB} = \dfrac{\Delta \vec{r}}{\Delta t} = \dfrac{\Delta x\ \hat{i} + \Delta y\ \hat{j}}{\Delta t}$

- Instantaneous velocity (at event A): $\vec{v}_A = \left.\dfrac{d\vec{r}}{dt}\right|_{t_A}$
  - Limit of average velocity as Δt approaches zero

- Speed – instantaneous magnitude of velocity: $v_A = \left|\dfrac{d\vec{r}}{dt}\right|_{t_A}$

Motion Path Example

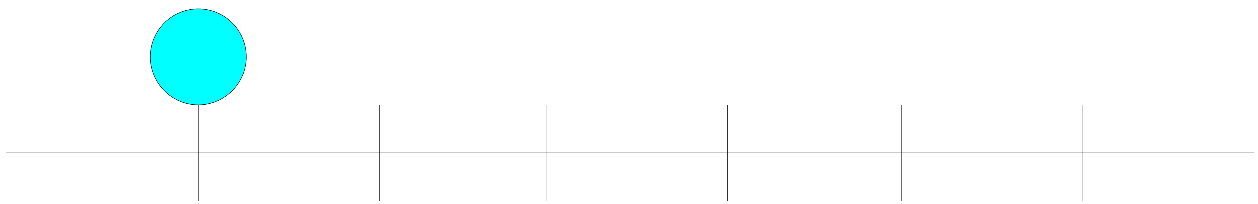
- 2-D motion path described by: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} B\,t^3 \\ C\,t^2 \end{pmatrix}$
  - What dimensions must B and C have?
  - Draw path on xy-plane

- Find instantaneous position, velocity, acceleration, jerk
  - At time t=0
  - At time t=1.0 second

- Find average velocity between t=3.0 sec and t=4.0 sec
  - Find average speed between t=3.0 sec and t=4.0 sec

Measuring Kinematic Quantities

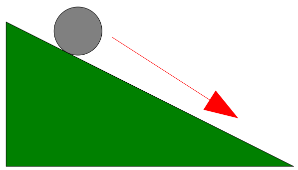
- Measuring average velocity:
  - Must measure position at beginning and end of time interval Δt



- Measuring instantaneous velocity:
  - Impossible to do directly! Why?
  - Must wait Δt to measure motion – instantaneous v may change
  - Instantaneous v can be measured indirectly
- Measurement of acceleration, etc. → similar issues
  - To measure directly → must measure instantaneous velocity

Motion with Constant Acceleration

- Early kinematic measurements (Galileo, Newton, etc.)
  - Produced results consistent with constant acceleration
  - Can be described by a Taylor series of 3 terms:



$$x(t) = x_0 + v_{x0}(t - t_0) + \frac{1}{2} a_x (t - t_0)^2$$

- x-component of velocity = derivative of x(t)

$$v_x(t) = \frac{dx}{dt} = v_{x0} + a_x (t - t_0)$$

- Algebra → can relate v_x and x directly (without t):

$$v_x^2 = v_{x0}^2 + 2 a_x (x - x_0)$$

Free Fall / Projectile Motion

- Experiments → gravity produces constant acceleration
  - Direction: straight down   Magnitude: $9.8 \frac{m}{sec^2} \equiv g$
  - g varies by a few percent over Earth surface

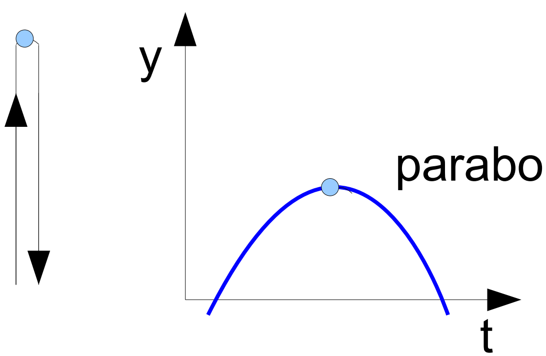
1-Dimensional Motion

Apply kinematic equations with:

$a_{vertical} = -g$

At "turning point" of motion:

$v_{vertical} = 0$



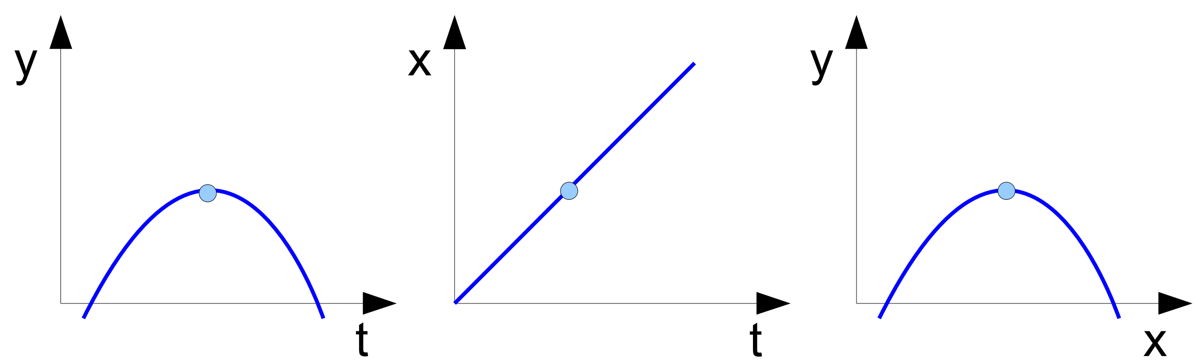
2-Dimensional Motion

Apply 2 sets of kinematic equations with:

$a_{vertical} = -g \qquad a_{horizontal} = 0$

At "turning point" of motion:

$v_{vertical} = 0 \qquad v_{horizontal} \neq 0$



Projectile Motion Example

- Quarterback can throw ball at speed $v_0 = 30$ m/sec
  - What is the maximum range he can throw down the field?
  - (Receiver and quarterback are same height, ignore air drag)
  - How many seconds is the ball in the air?

- What is the maximum range if:
  - Quarterback jumps and throws (from a height of 3.0 meters)
  - Receiver dives for the catch (at a height of 0.0 meters)

Varying Acceleration: The Chain Rule

- Definition of x-component of acceleration: $$a_x = \frac{d^2x}{dt^2} = \frac{dv_x}{dt}$$

- Apply the chain rule: $$a_x = \frac{dv_x}{dt} = \frac{dx}{dt}\frac{dv_x}{dx} = v_x \frac{dv_x}{dx}$$
  - Can relate x, v_x, and a_x without using t
  - Useful if a_x can be expressed as function of x or v_x (but not t)

- If a_x is a function of x: $$a_x\, dx = v_x\, dv_x \longrightarrow \int a_x\, dx = \frac{1}{2}\left(v_x^{\,2} - v_{x0}^{\,2}\right)$$
  - Examples: springs, inclines of non-constant slope

- If a_x is a function of v_x: $$dx = \frac{v_x\, dv_x}{a_x} \longrightarrow x - x_0 = \int \left(\frac{v_x}{a_x}\right) dv_x$$
  - Examples: air resistance, viscous drag

Chain Rule Example

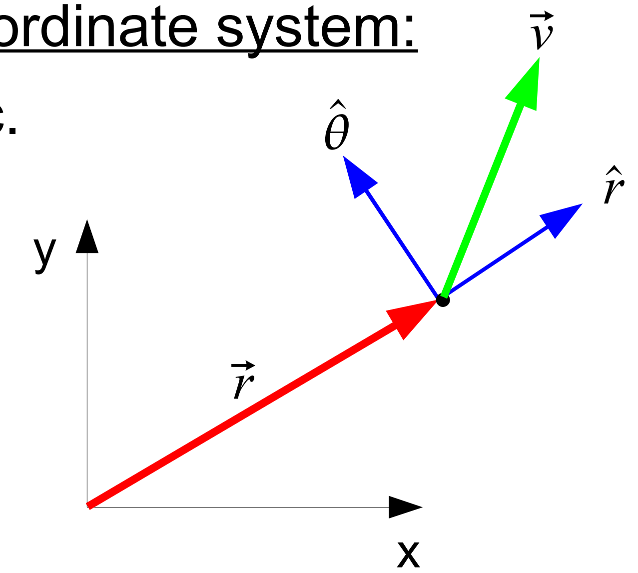
- One mathematical model for air resistance: $\vec{a} = -k\ \vec{v}$

- For 1-D motion with a given k, x_0, and v_{x0}, calculate:
  - v_x as a function of $(x - x_0)$
  - a_x as a function of $(x - x_0)$
  - v_x as a function of $(t - t_0)$

- For 2-D motion with gravity included, calculate:
  - $x(t - t_0)$ and $y(t - t_0)$ – given x_0, y_0, v_{x0}, v_{y0}

Kinematics in Polar Coordinates

- Position, velocity, acceleration, jerk, etc. → all vectors
  - Components can be expressed in any coordinate system:
  - Cartesian, polar, cylindrical, spherical, etc.



- In 2-D polar coordinates: $\vec{r} = r\ \hat{r}$

$$\hat{r} = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$$

$$\hat{\theta} = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j}$$

- Notation – "dot on top" means time derivative → Using the chain rule:

$$\frac{d\ \hat{r}}{dt} = -\sin\theta\ \dot{\theta}\ \hat{i} + \cos\theta\ \dot{\theta}\ \hat{j} = \dot{\theta}\ \hat{\theta}$$

$$\vec{v} = \frac{d\ \vec{r}}{dt} = \frac{d}{dt}\left(r\ \hat{r}\right)$$

$$\frac{d\ \hat{\theta}}{dt} = -\cos\theta\ \dot{\theta}\ \hat{i} - \sin\theta\ \dot{\theta}\ \hat{j} = -\dot{\theta}\ \hat{r}$$

$$\vec{v} = \dot{r}\ \hat{r} + r\ \dot{\theta}\ \hat{\theta}$$

Kinematics in Polar Coordinates

- Acceleration in 2-D Polar Coordinates:

$$\vec{a} = \frac{d\,\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\hat{r} \;+\; r\,\dot{\theta}\,\hat{\theta}\right)$$

$$\vec{a} = \left(\ddot{r}\,\hat{r} + \dot{r}\,\dot{\theta}\,\hat{\theta}\right) \;+\; \left(\dot{r}\,\dot{\theta}\,\hat{\theta} + r\,\ddot{\theta}\,\hat{\theta} + r\,\dot{\theta}\left(-\dot{\theta}\,\hat{r}\right)\right)$$

$$\vec{a} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{r} \;+\; \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\hat{\theta}$$

"centripetal" acceleration

"Coriolis" acceleration – note: velocity-dependent

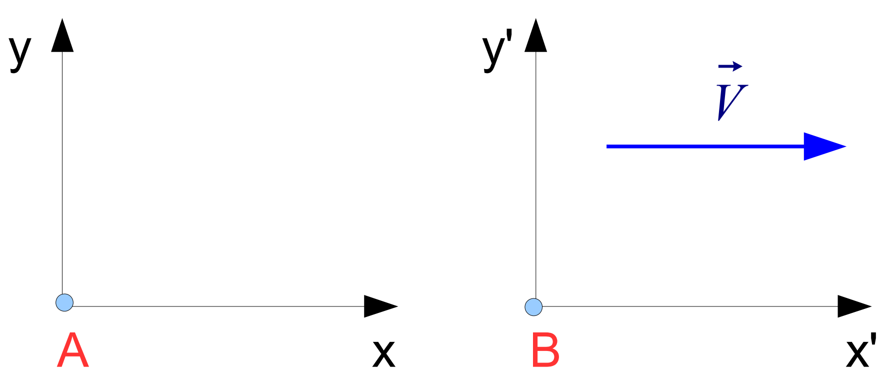
- Example: Motion Path $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C \\ Dt \end{pmatrix}$

  – Calculate polar components of position, velocity, acceleration

“Newtonian Relativity”

- For a vector space with time included as a coordinate:
  - Any event can be chosen as the origin (“homogeneous”)
  - Any direction can be chosen for coordinate axes (“isotropic”)
  - Including directions with nonzero projection onto “time axis”

- 2 different choices of origin and coordinate axes:
  - 2 reference frames with relative velocity between them
  - Physics needs to be able to handle any choice of frame



Event A is at the origin of both reference frames at $t' = t = 0$

Event B occurs at time $Vt' = Vt = 1$ meter

“primed” frame: $(Vt', x', y', z') = (1, 0, 0, 0)$

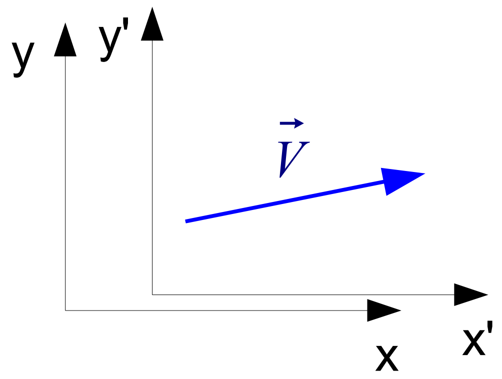
“unprimed” frame: $(Vt, x, y, z) = (1, 1, 0, 0)$

Galilean Transformations

- Physics must work equally well in any reference frame
  - With appropriate coordinate transformations relating frames
  - Includes any 2 frames in relative motion (translation or rotation)

- Newtonian Physics → use "Galilean Transformations":
  - For simplicity, assume the event (0,0,0,0) is same in both frames

- Translation:



For a given event:

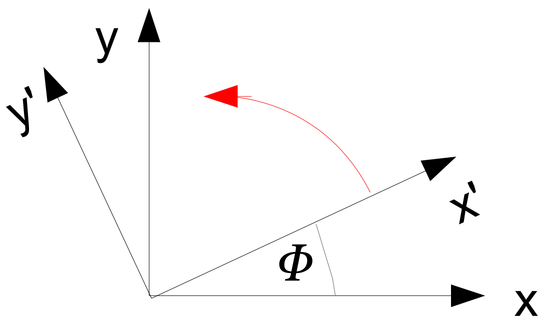
$$t' = t$$
$$\vec{r}\,'(t) = \vec{r}(t) - \vec{R}(t)$$
$$\vec{v}\,'(t) = \vec{v}(t) - \vec{V}(t)$$
$$\vec{a}\,'(t) = \vec{a}(t) - \vec{A}(t)$$

If **V** = constant:

$$t' = t$$
$$\vec{r}\,'(t) = \vec{r}(t) - \vec{V}\,t$$
$$\vec{v}\,'(t) = \vec{v}(t) - \vec{V}$$
$$\vec{a}\,'(t) = \vec{a}(t)$$

Rotating Reference Frames

- Rotation (about z-axis) – Φ now a function of time:



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R(\Phi)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\Phi + y\sin\Phi \\ -x\sin\Phi + y\cos\Phi \end{pmatrix}$$

- Using the chain rule: $\begin{pmatrix} v_x' \\ v_y' \end{pmatrix} = \frac{d}{dt}\left(R(\Phi)\begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\frac{dR(\Phi)}{dt}\begin{pmatrix} x \\ y \end{pmatrix} + R(\Phi)\begin{pmatrix} v_x \\ v_y \end{pmatrix}\right)$

$$\begin{pmatrix} a_x' \\ a_y' \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} v_x' \\ v_y' \end{pmatrix} = \left(\frac{d^2R(\Phi)}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} + 2\frac{dR(\Phi)}{dt}\begin{pmatrix} v_x \\ v_y \end{pmatrix} + R(\Phi)\begin{pmatrix} a_x \\ a_y \end{pmatrix}\right)$$

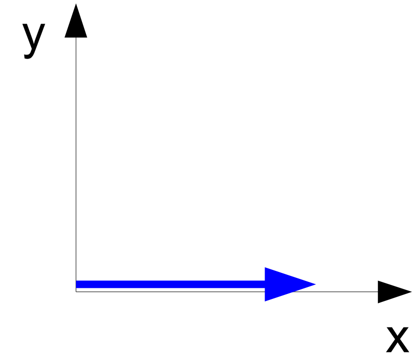
$$\vec{a}' = -\dot{\Phi}^2 R(\Phi)\,\vec{r} + 2\frac{d\,R(\Phi)}{dt}\vec{v} + R(\Phi)\,\vec{a}$$

Acceleration in a rotating frame (S') depends on:

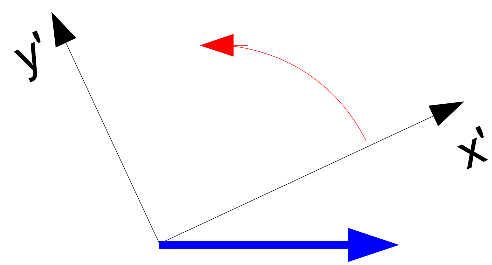
- Position, velocity, and acceleration in frame S
- Contrast this with translating frames (Galilean Transformation)

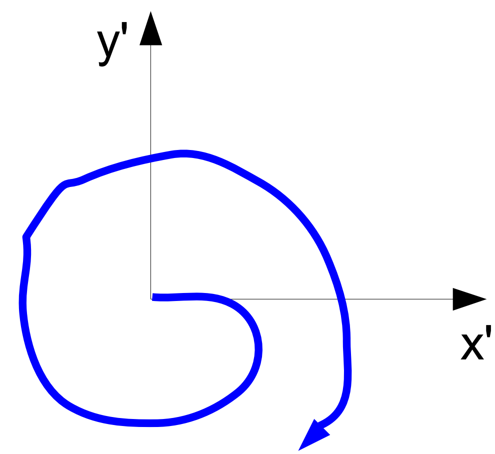
Rotating Reference Frame Example

- Consider a motion path in frame S
  - Moving along x-axis with constant speed v



- S' → Frame rotating at constant angular rate ω
  - Calculate $x'(t)$, $y'(t)$, $a'_{x'}(t)$, $a'_{y'}(t)$ in S'





- In S', path is curved → requires acceleration
  - S/S' will measure different a – even with constant rotation rate
  - Physics → still consistent if a can be <u>transformed</u> consistently