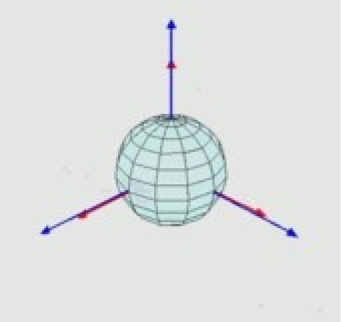
ROTATIONAL KINEMATICS

Angular Velocity Vector



- Consider a rotation which evolves over time:

$$\hat{x}'(t),\ \hat{y}'(t),\ \hat{z}'(t) \quad \text{or} \quad R(t) = \begin{pmatrix} R_{xx}(t) & R_{xy}(t) & R_{xz}(t) \\ R_{yx}(t) & R_{yy}(t) & R_{yz}(t) \\ R_{zx}(t) & R_{zy}(t) & R_{zz}(t) \end{pmatrix}$$

  - Both axis and rate of rotation may change over time

- Over infinitesimal time interval between *t* and *t+dt*:
  - Rotation about an "instantaneous axis" by small angle $d\theta$:
  - $R_{\vec{A}}(d\theta) = R_x(d\theta_x)\, R_y(d\theta_y)\, R_z(d\theta_z)$ (in any order)
  - Can write rate of rotation as a vector – xyz treated equally

- Angular velocity vector:

$$\boxed{\vec{\Omega} \equiv \frac{d\theta_x}{dt}\hat{i} + \frac{d\theta_y}{dt}\hat{j} + \frac{d\theta_z}{dt}\hat{k}}$$

(Points along axis of rotation)

Rotational Kinematics

- Taylor Series for angular velocity:

$$\vec{\Omega} = \vec{\Omega}_0 + \left.\frac{d\vec{\Omega}}{dt}\right|_{t_0} (t - t_0) + \frac{1}{2} \left.\frac{d^2\vec{\Omega}}{dt^2}\right|_{t_0} (t - t_0)^2 + \ldots$$

"angular acceleration" ($\vec{\alpha}$) – may or may not point along $\vec{\Omega}$

  - Similar to translational kinematics, with no "position vector"

- For rotations about a constant axis:
  - Rotations do commute → can assign an "angular position" θ
  - Taylor Series for rotation angle (about a constant axis only):

$$\theta = \theta_0 + \underbrace{\left.\frac{d\theta}{dt}\right|_{t_0}}_{\Omega_0} (t - t_0) + \frac{1}{2} \underbrace{\left.\frac{d^2\theta}{dt^2}\right|_{t_0}}_{\alpha_0} (t - t_0)^2 + \ldots$$

1-D kinematics equations for x, v_x, a_x can now be applied to θ, Ω, α

Time Derivative in Rotating Frames

- R matrix transforms components of a vector: $\vec{u}\,' = R\,\vec{u}$
  - Time derivatives in rotating frames must take into account:
  - 1) time dependence of the actual vector
  - 2) changing direction of coordinate axes→time dependence of R

- Example: Frame S' rotating with angular velocity $\vec{\Omega}$
  - Using the chain rule: $\dfrac{d\,\vec{u}\,'}{dt} = \left(R_{\vec{\Omega}}\right)\dfrac{d\,\vec{u}}{dt} + \left(\dfrac{d\,R_{\vec{\Omega}}}{dt}\right)\vec{u}$

- By definition: $\dfrac{dR_{\vec{\Omega}}}{dt} = \dfrac{R_{\vec{\Omega}}(t+dt) - R_{\vec{\Omega}}(t)}{dt} = \left(\dfrac{R_{\vec{\Omega}}(\Omega\ dt) - 1}{dt}\right)R_{\vec{\Omega}}(t)$

Plug in $R_{\vec{\Omega}}(\Omega\ dt) = \begin{pmatrix} 1 & \Omega_z\,dt & -\Omega_y\,dt \\ -\Omega_z\,dt & 1 & \Omega_x\,dt \\ \Omega_y\,dt & -\Omega_x\,dt & 1 \end{pmatrix} \longrightarrow \boxed{\dfrac{dR_{\vec{\Omega}}}{dt} = \begin{pmatrix} 0 & \Omega_z & -\Omega_y \\ -\Omega_z & 0 & \Omega_x \\ \Omega_y & -\Omega_x & 0 \end{pmatrix} R_{\vec{\Omega}}}$

Time Derivative in Rotating Frames

- Examine the effect of matrix $\begin{pmatrix} 0 & \Omega_z & -\Omega_y \\ -\Omega_z & 0 & \Omega_x \\ \Omega_y & -\Omega_x & 0 \end{pmatrix}$ on a vector

$$\begin{pmatrix} 0 & \Omega_z & -\Omega_y \\ -\Omega_z & 0 & \Omega_x \\ \Omega_y & -\Omega_x & 0 \end{pmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} \Omega_z\, b_y - \Omega_y\, b_z \\ \Omega_x\, b_z - \Omega_z\, b_x \\ \Omega_y\, b_x - \Omega_x\, b_y \end{pmatrix} = \boxed{-\vec{\Omega} \times \vec{b}}$$

The cross product can be written as an operator in matrix form!

- Time derivative of a vector in a rotating frame:

$$\frac{d\,\vec{u}\,'}{dt} = R\,\frac{d\,\vec{u}}{dt} - \left[\vec{\Omega} \times \left(R\,\vec{u}\right)\right] \longrightarrow$$

Vectors which are defined by a time derivative (e.g. velocity) pick up an extra term when they are transformed

- Examples: Velocity → $\boxed{\vec{v}\,' = R\,\vec{v} - \left[\vec{\Omega} \times \left(R\,\vec{r}\right)\right]}$

- Acceleration

$$\left\{ \begin{array}{l} \dfrac{d\,\vec{v}\,'}{dt} = R\,\dfrac{d\,\vec{v}}{dt} + \dfrac{dR}{dt}\,\vec{v} - \left[\vec{\Omega} \times \left(R\,\dfrac{d\,\vec{r}}{dt} + \dfrac{dR}{dt}\,\vec{r}\right)\right] \\[2ex] \boxed{\vec{a}\,' = R\,\vec{a} - 2\left[\vec{\Omega} \times R\,\vec{v}\right] + \left[\vec{\Omega} \times \left(\vec{\Omega} \times R\,\vec{r}\right)\right]} \end{array} \right.$$

Coriolis and Centrifugal Acceleration

$$\vec{a}\,' = R\,\vec{a} - 2\left[\vec{\Omega}\times R\,\vec{v}\right] + \left[\vec{\Omega}\times\left(\vec{\Omega}\times R\,\vec{r}\right)\right]$$

- More useful to relate **a'** to **v'** and **r'** (instead of **v** and **r**)
  - This way, all measurements can be made in rotating frame

Plug in $R\,\vec{r} = \vec{r}\,'$

and $R\,\vec{v} = \vec{v}\,' + \left[\vec{\Omega}\times\vec{r}\,'\right]$

$$\vec{a}\,' = R\,\vec{a} - 2\left[\vec{\Omega}\times\left(\vec{v}\,' + \left(\vec{\Omega}\times\vec{r}\,'\right)\right)\right] + \left[\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\,'\right)\right]$$

$$\boxed{\vec{a}\,' = R\,\vec{a} - 2\left[\vec{\Omega}\times\vec{v}\,'\right] - \left[\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\,'\right)\right]}$$

Inertial acceleration

Coriolis acceleration

Centrifugal acceleration

$\mathbf{a}_{\text{centrifugal}}$ (Index Notation)

$$\vec{a}_{centrifugal} = -\left[\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right] = -\,\Omega_i\,\hat{e}_i\times\left(\Omega_j\,\hat{e}_j\times r_k\,\hat{e}_k\right)$$

$$\vec{a}_{centrifugal} = -\,\Omega_i\,\hat{e}_i\times\left(\Omega_j\,r_k\,\epsilon_{jkl}\,\hat{e}_l\right) = -\,\Omega_i\left(\Omega_j\,r_k\,\epsilon_{jkl}\right)\left(\epsilon_{ilm}\,\hat{e}_m\right)$$

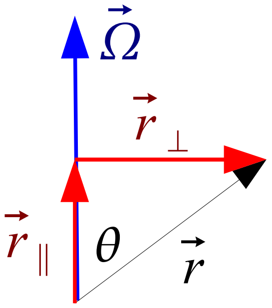
$$\vec{a}_{centrifugal} = \left(-\epsilon_{jkl}\,\epsilon_{mil}\right)\Omega_i\,\Omega_j\,r_k\,\hat{e}_m = \left(\delta_{ji}\,\delta_{km}-\delta_{jm}\,\delta_{ki}\right)\Omega_i\,\Omega_j\,r_k\,\hat{e}_m$$

$$\vec{a}_{centrifugal} = \Omega_i\,\Omega_i\,r_k\,\hat{e}_k \;-\; \Omega_i\,\Omega_j\,r_i\,\hat{e}_j$$

$$\vec{a}_{centrifugal} = \Omega^2\,\vec{r} \;-\; \left(\vec{\Omega}\cdot\vec{r}\right)\vec{\Omega}$$

$$\vec{a}_{centrifugal} = \Omega^2\,\vec{r} - \left(\Omega\;r\;\cos\,\theta\right)\left(\Omega\;\hat{\Omega}\right)$$

$$\vec{a}_{centrifugal} = \Omega^2\left(\vec{r}-\vec{r}_{\parallel}\right) = \boxed{\Omega^2\,\vec{r}_{\perp}}$$



- $\mathbf{a}_{\text{centrifugal}}$ always points directly away from rotation axis

Coriolis / Centrifugal Examples

- Bead on spoke of bicycle wheel rotating at Ω
  - Slides outward such that $r(\theta) = r_0 + k(\theta - \theta_0)$ (ignore gravity)
  - Calculate r'(t), v'(t), Coriolis and centrifugal accelerations

- Spacecraft approaches S.B. from above (34° latitude)
  - At some instant:
  - Earth rotates at angular velocity Ω, spacecraft is at r = $2R_E$
  - Falling straight down at speed v (in inertial space)
  - Calculate (North/South, East/West, Up/Down) components of v'
  - Calculate components of Coriolis and centrifugal accelerations

Fictitious vs. Real Forces

- Normal, Friction, Tension, Compression
  - Exerted by particles on other particles nearby
  - Newtonian Physics does not attempt to describe further

- Does a particle exert Coriolis and centrifugal forces?
  - No! They are a property of the rotating space itself

- Which type of force is gravity? Has aspects of both...
  - Gravitational force is directly exerted by space
  - But has particles as its source
  - Einstein's work shows gravity acts more like a fictitious force
  - Examples: 1) Does gravity affect light? 2) orbit of Mercury