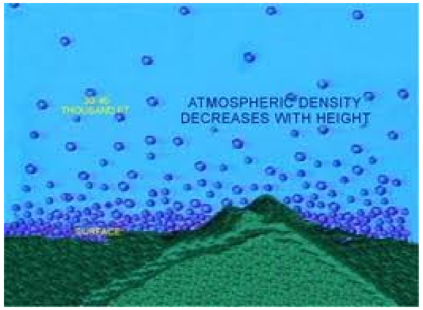
SOUND WAVES

Mathematical Models of Fluids

- Fluids → molecules roam and collide – no “springs”
  - Collisions cause pressure in fluid (Units: Pascal – Pa = N/m^2)

- 2 mathematical models for fluid motion:
  - 1) “Bulk” properties – pressure/density at each point
  - 2) Statistical treatment of molecules – velocity/temperature

- Equilibrium state: uniform pressure/density: $p(x, y, z, t) = p_0$, $\rho(x, y, z, t) = \rho_0$
  - External forces can cause “pressure gradient”
  - Example: uniform gravity acting on air
  - Assume $p = C\rho$ → calculate $p(z)$ and $\rho(z)$

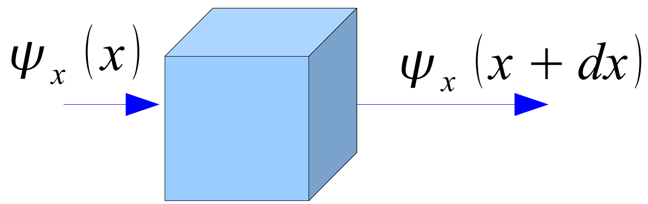


  - In general, force due to non-uniform pressure: $d\vec{F}_p = -(\vec{\nabla} p)\, dV$

Pressure and Displacement in Fluids

- Force on fluid → displacement and pressure gradient
  - Example: firecracker – air is pushed outward for an instant
  - Sketch graphs of ψ_r vs. r and p vs. r just after explosion

- Recall definition of Bulk Modulus: $$B = -V\frac{\Delta p}{\Delta V}$$



This parcel of fluid is "stretched" (dV > 0) due to uneven Ψ

Note: dV for a "parcel" of fluid depends on the difference in Ψ from one side of parcel to other

$$\Delta V = (\Delta\psi_x)\,dy\,dz + (\Delta\psi_y)\,dx\,dz + (\Delta\psi_z)\,dx\,dy$$

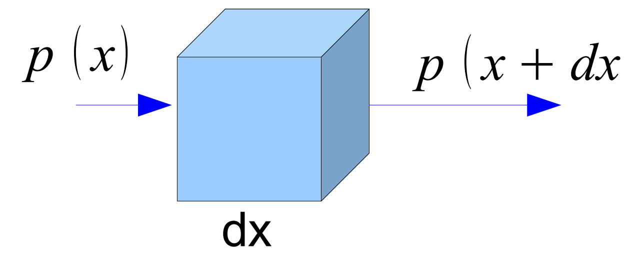
$$\Delta V = \left(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y} + \frac{\partial\psi_z}{\partial z}\right)dx\,dy\,dz$$

$$\frac{\Delta V}{V} = \frac{\Delta V}{dx\,dy\,dz} = \vec{\nabla}\cdot\vec{\psi}$$

$$\Delta p = -B\left(\vec{\nabla}\cdot\vec{\psi}\right)$$

Sound Waves in Fluids

- Newton's 2nd Law: $d\,\vec{F} = dm\,\vec{a}$



$$-\left(\vec{\nabla}\,p\right) dV = \left(\rho\,dV\right)\left(\frac{\partial^2\,\vec{\psi}}{\partial\,t^2}\right)$$

Cancel dV and take divergence of both sides:

$$\vec{\nabla}\cdot\left(-\vec{\nabla}\,p\right) = \vec{\nabla}\cdot\left(\rho\,\frac{\partial^2\,\vec{\psi}}{\partial\,t^2}\right)$$

$$-\nabla^2\,p = \rho\,\frac{\partial^2\left(\vec{\nabla}\cdot\vec{\psi}\right)}{\partial\,t^2} = \rho\,\frac{\partial^2}{\partial\,t^2}\,\frac{p}{B}$$

Plug in from the definition of bulk modulus:

$$\nabla^2\,p = \frac{\rho}{B}\,\frac{\partial^2\,p}{\partial\,t^2}$$

This is the 3-D wave equation using pressure as the variable

$$v = \sqrt{\frac{B}{\rho}}$$

- Pressure of neighboring parcels of fluid
  - Has similar physical effect to “springs” – restoring force
  - Leads to waves: $P = p_0 + p\left(k_x\,x + k_y\,y + k_z\,z \pm \omega\,t\right)$

Transforming Derivatives

- In Cartesian coordinates: $\nabla^2 p = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) p$
  - Commonly, sound waves expand from a point or line source
  - How to express this in spherical or cylindrical coordinates?

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy + \frac{\partial p}{\partial z} dz = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial \theta} d\theta + \frac{\partial p}{\partial \phi} d\phi$$

To transform derivatives to a different coordinate system:

Must express dr, dθ, dϕ in terms of dx, dy, dz

Then plug in to 3-D wave equation above

Spherical – Partial derivatives expressed in matrix form:

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

$$\begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{-\cos\theta\cos\phi}{r} & \frac{-\cos\theta\sin\phi}{r} & \frac{\sin\theta}{r} \\ \frac{-\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

3-D Wave Equation

$$\left(\frac{\partial}{\partial x}\right) p = \left(\sin\theta\cos\phi\frac{\partial}{\partial r} - \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) p$$

- Repeat for y,z → plug in and use chain rule
  - To get full 3-D wave equation in spherical coordinates

- Common simplifying assumptions:
  - Spherical symmetry (i.e. wave is isotropic: $\frac{\partial p}{\partial\theta} = \frac{\partial p}{\partial\phi} = 0$)
  - In this case: $\nabla^2 p = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right)$
  - Cylindrical symmetry (i.e. “line” source: $\frac{\partial p}{\partial\phi} = \frac{\partial p}{\partial z} = 0$)
  - In this case: $\nabla^2 p = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial p}{\partial\rho}\right)$

Sound Wave Intensity

- Sound wave – at any given instant:
  - PE stored in pressure gradient (i.e. non-uniform pressure)
  - KE stored in motion of fluid
  - Energy travels in a particular direction
  - "Direction of propagation" – determined by k_x, k_y, k_z

- Wave Intensity (Units: W/m^2)
  - Average power per unit area transmitted by wave
  - Area must be measured perpendicular to propagation
  - For linear waves: Intensity is proportional to $(\text{amplitude})^2$
  - Human ear can detect incredibly wide range of intensities
  - Quietest: 10^{-12} W/m^2 Loudest (before injury): 10 W/m^2

Examples

- Consider a traveling longitudinal "plane wave":

$$p(x, y, z, t) = p_0 \cos(k_x x - \omega t)$$

  - Given the bulk modulus B and density ρ:
  - Calculate the displacement field in terms of B, ρ, p_0, k_x
  - Calculate the wave intensity in terms of B, ρ, p_0, k_x
  - How might one physically create a plane wave?

- Consider an outgoing spherical wave:

$$p(r, t) = p_0 \left(\frac{r_0}{r}\right) \sin(kr - \omega t)$$

  - Show that this satisfies wave equation in spherical coordinates
  - Calculate the wave intensity as a function of r

Decibel Scale

- Intensity / frequency ranges of human hearing: wide
  - Loudest vs. quietest sound: 12 orders of magnitude in intensity
  - But perceived loudness is not "a trillion times louder"

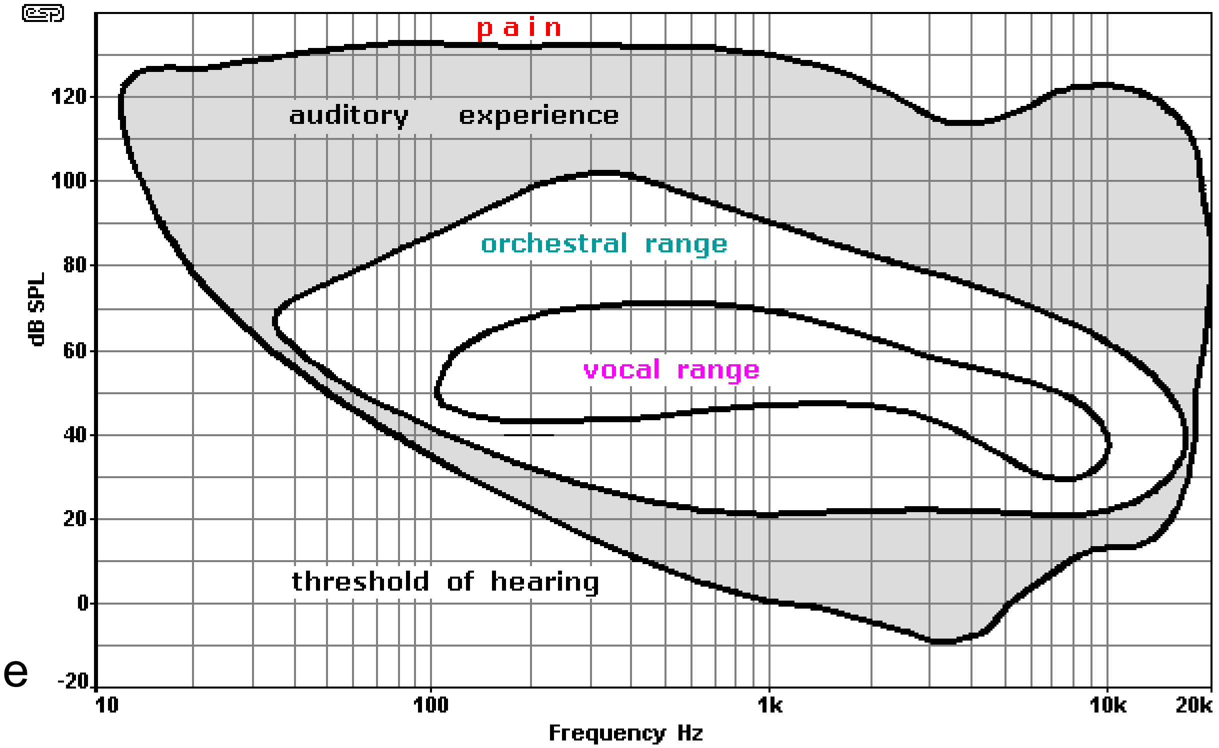
- Decibel Scale – designed based on human perception

$$\beta = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

$$I_0 = 10^{-12} \frac{W}{m^2}$$

Logarithmic scale:

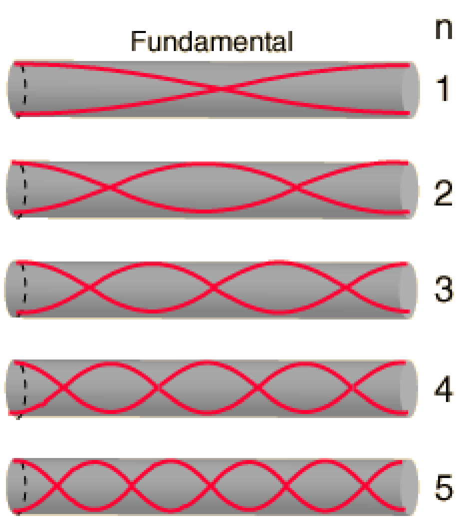
"Compresses" wide range of numbers to more manageable range



Wind Instruments

- 3-D sound waves can be reduced to 1-D
  - By use of a tube or pipe → standing waves inside
  - These "wind instruments" have similar physics to strings
  - <u>Frequency of standing waves depends on:</u>
  - 1) length of tube 2) whether ends are open or closed

Both ends open:



$$\lambda_n = \frac{2L}{n}$$

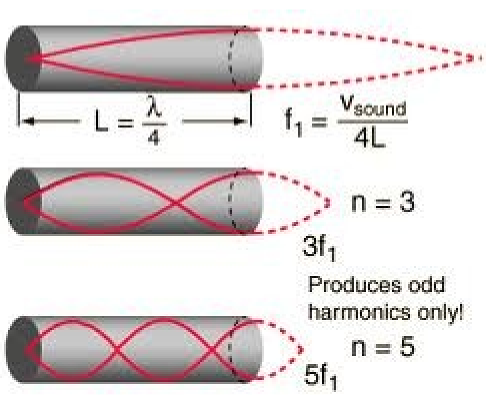
$$f_n = n\, f_1$$

(exactly like strings)

One end closed:

$$\lambda_n = \frac{2L}{n + \frac{1}{2}}$$

$$f_n = (2n + 1)\, f_1$$

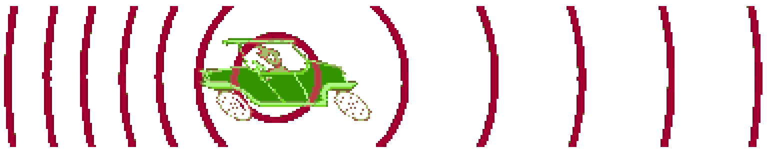


<u>Examples</u>: reed-based instruments, some organs

Doppler Effect

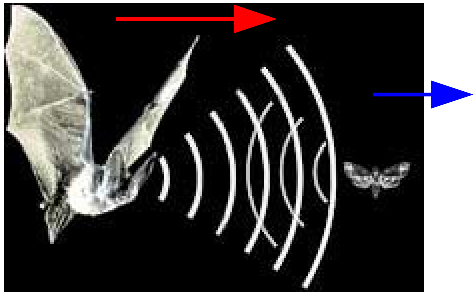
- Relative velocity of sound source/listener:
  - Perceived frequency/wavelength differs from source
  - Relative motion toward each other → $f_L > f_S$

$$f_L = \left(\frac{v \pm v_L}{v \pm v_S}\right) f_S$$



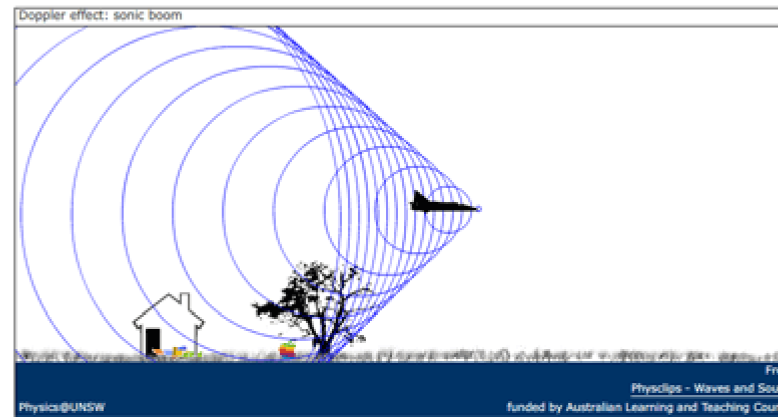
  - Note: v_L and v_S must be measured relative to medium
  - i.e. there is a "preferred" reference frame for sound waves

- Example: bat chasing moth – uses "sonar"
  - Given v_{bat} and v_{moth} relative to air:
  - Calculate ratio $f_{perceived}$ / $f_{emitted}$
  - Make sure to pick the correct signs!

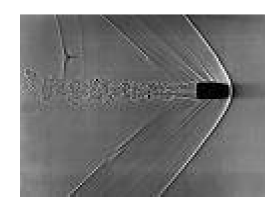


Shock Waves

- Problem with Doppler shift formula:
  - What happens as v_S approaches speed of sound?
  - Sound waves do not "escape" source quickly
  - Many waves build up in small region in front of source:



- Shock wave (nonlinear)
  - Small area of <u>very</u> high intensity
  - Produced when linear waves can not move energy fast enough



- Examples:
  - Supersonic jet, bullet → conical shock wave → sonic boom
  - Atomic bomb → spherical shock wave → huge impact force

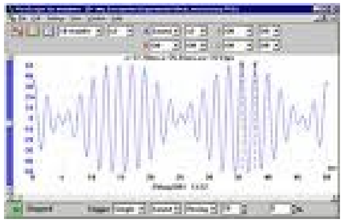
Beats

- Consider the combination of 2 pure tones:

$$p(t) = p_0 \cos(\omega_1 t) + p_0 \cos(\omega_2 t)$$

  - Plug in the trig identity $\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

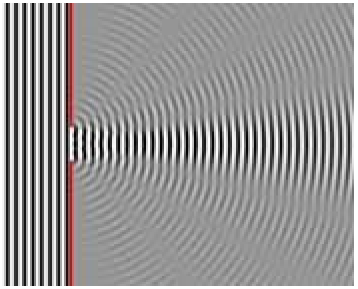
$$p(t) = \left[2\, p_0 \cos\left(\left(\frac{\omega_1 - \omega_2}{2}\right) t\right)\right] \cos\left(\left(\frac{\omega_1 + \omega_2}{2}\right) t\right)$$



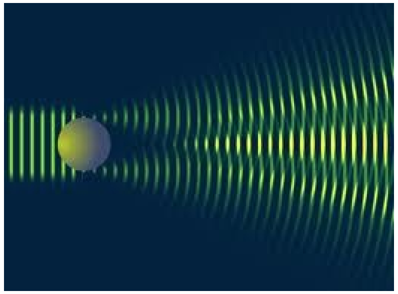
- Single tone → frequency = average of 2 pure tones
  - Amplitude varies → frequency = half difference of 2 pure tones
  - Perceived sound: single tone with "wah-wah-wah"

- "Beat frequency" defined as $|f_1 - f_2|$ → uses:
  - Tuning musical instruments → adjust until beats vanish
  - Detection of low speeds using Doppler reflection

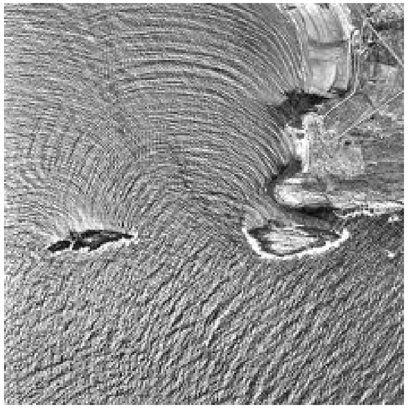
Diffraction



- Consider waves at the "edge" of a boundary:
  - Can a "plane wave" be confined to a small area?
- Wave equation does not allow for abrupt wave "edges"
  - If $\nabla^2 p \neq 0$ then $\frac{\partial^2 p}{\partial t^2} \neq 0$ $\rightarrow$ wave "spreads out"



  - Also occurs when waves encounter obstacles
- Diffraction – highly dependent on wavelength



  - Long wavelengths "bend" more easily
  - Short wavelengths → more "directional"
  - Example: Bass sound tends to "fill" a room