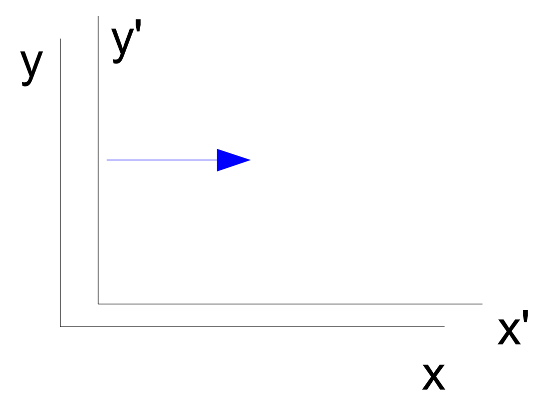
SPECIAL RELATIVITY

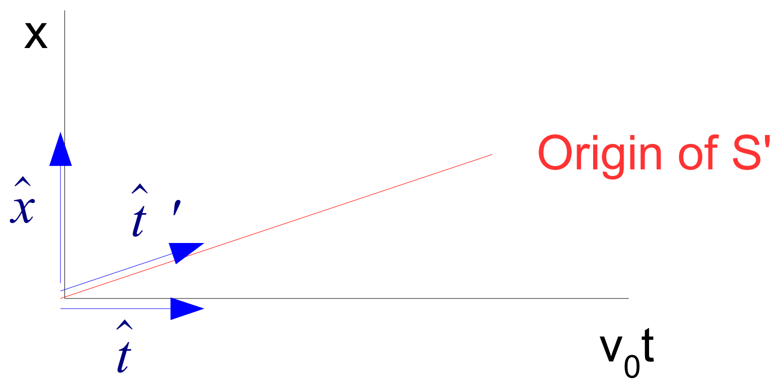
The "Time Coordinate"

- Newtonian point particles – described by $x(t)\,,\; y(t)\,,\; z(t)$
  - t is a "parameter" relating particle's xyz to each other
  - Particle motion determined by masses and forces
- Fields / waves / mass distributions: $\psi(x\,,\, y\,,\, z\,,\, t)$
  - Mathematically: x,y,z,t treated on equal footing
  - Example: $\psi(x\,,\, y\,,\, z\,,\, t) \;=\; f\,(k_x\, x \;+\; k_y\, y \;+\; k_z\, z \;+\; k_t\,(v_0\, t))$
  - $v_0 t$ is a 4th coordinate rather than a parameter (v_0 is arbitrary)
- Particles in 4-D "spacetime": $x(\tau)\,,\; y(\tau)\,,\; z(\tau)\,,\; v_0\, t(\tau)$
  - Where τ is a new parameter relating x,y,z,(v_0t) to each other
  - How to decide between 3-D and 4-D views? – experiment

“Direction” of Time

- v_0t is a coordinate → should have a “unit vector”
  - In a given reference frame S: $\hat{t} \perp \hat{x}, \hat{y}, \hat{z}$
  - Particle sitting still – moving in “time direction” only
- Consider a reference frame S' (at relative speed V):
  - By definition: $\hat{t} \perp \hat{x}$ $\qquad \hat{t}' \perp \hat{x}'$
  - Are these true? $\hat{t}' \parallel \hat{t}$ $\qquad \hat{t}' \perp \hat{x}$





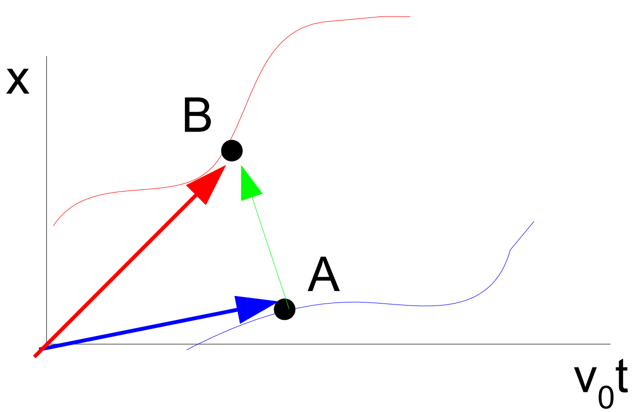
In 4-D spacetime, time's “direction” depends on the reference frame

4-D Spacetime Geometry

- Made up of "events" (rather than points)
  - Each event has $x,y,z,(v_0t)$ coordinates

$$\tilde{X} \equiv \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} v_0 t \\ x \\ y \\ z \end{pmatrix}$$

- Each event defined by a position "4-vector"
  - Relative to the origin (0,0,0,0) of the reference frame
  - Convention: time component is first entry, given index 0



Event A

$$\tilde{X}_A = \begin{pmatrix} v_0 t_A \\ x_A \\ 0 \\ 0 \end{pmatrix}$$

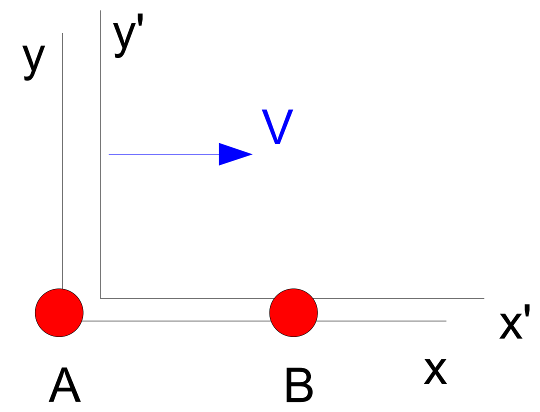
Event B

$$\tilde{X}_B = \begin{pmatrix} v_0 t_B \\ x_B \\ 0 \\ 0 \end{pmatrix}$$

Displacement 4-vector:

$$\tilde{S}_{AB} = \begin{pmatrix} v_0 t_B - v_0 t_A \\ x_B - x_A \\ 0 \\ 0 \end{pmatrix}$$

Spacetime Example



- Consider 4-D frames S and S':
- Event A: spatial origin of S and S' meet at $t = t' = 0$
- Event B: spatial origin of S' is at $x=D$ at time $t=T$
- Write position 4-vector for each event in each frame
  - Assume (for now) that $t'=t$ → "Galilean Transformation"
  - Calculate displacement 4-vector (A→B) in each frame
  - What is the 4-D "length" of displacement vector in each frame?

Galilean Transformation in Spacetime

- Galilean Transformation in 4-D matrix form:
  - an assumption of classical physics!

$$\begin{pmatrix} x_0{}' \\ x_1{}' \\ x_2{}' \\ x_3{}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \left(\frac{-V}{v_0}\right) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

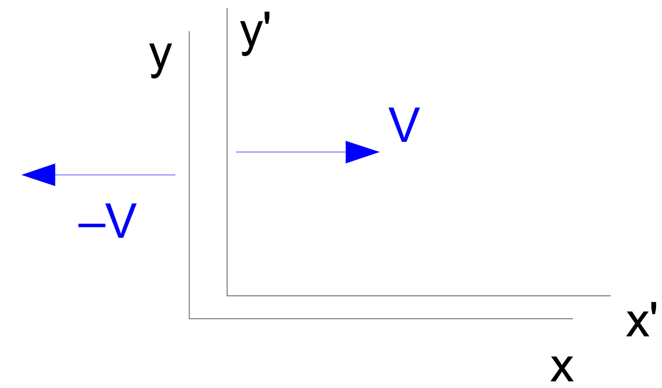
- Does not agree with 4-D geometry:
  - Changes "length" of displacement 4-vector between 2 events
  - Does not treat x_0 and x_1 on equal footing (no matrix symmetry)
  - Goal: Find a matrix which keeps length of 4-vector invariant
  - i.e. a 4-D "rotation" matrix from S to S'

4-D "Rotation" Matrix

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Where A,B,C,D are functions of V

By switching V to –V
→ get inverse matrix



Length of 4-vector invariant: $$(A\ x_0 + B\ x_1)^2 + (C\ x_0 + D\ x_1)^2 = x_0^2 + x_1^2$$

$$(A^2 + C^2)\ x_0^2 + (C^2 + D^2)\ x_1^2 + (2\ A\ B + 2\ C\ D)\ x_0\ x_1 = x_0^2 + x_1^2$$

$$A^2 + C^2 = 1$$

$$B^2 + D^2 = 1$$

$$2\ A\ B + 2\ C\ D = 0$$

$$D = \pm A$$

$$B = \pm C = \pm\sqrt{1 - A^2}$$

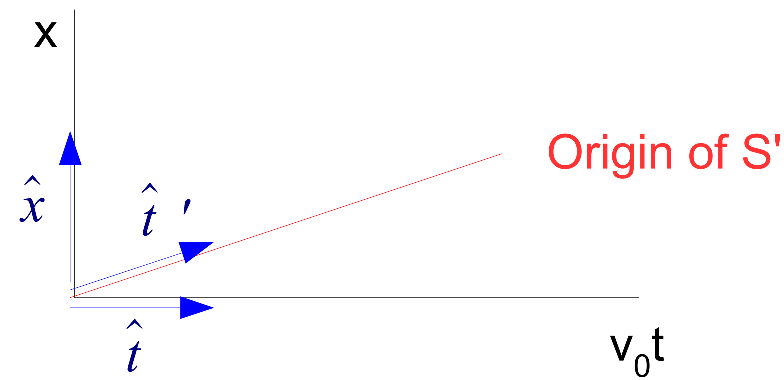
What do +– physically represent?

$$\begin{pmatrix} A & \pm\sqrt{1-A^2} & 0 & 0 \\ \pm\sqrt{1-A^2} & A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4-D "Rotation" Matrix

- Invariance of 4-vector length → 3 eqns., 4 unknowns
  - Not enough information to solve for A,B,C,D
  - More info needed → consider motion of S' origin in frame S:

$$\begin{pmatrix} v_0 t' \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A & \pm\sqrt{1-A^2} & 0 & 0 \\ \pm\sqrt{1-A^2} & A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_0 t \\ V t \\ 0 \\ 0 \end{pmatrix}$$



x_1 component: $0 = \pm\sqrt{1-A^2}\, v_0 t + A V t$

$$\pm\sqrt{1-A^2} = -\frac{V}{v_0} A$$

$$A = \frac{1}{\sqrt{1+\frac{V^2}{v_0^2}}}$$

$$R = \begin{pmatrix} \frac{1}{\sqrt{1+(V/v_0)^2}} & \frac{(V/v_0)}{\sqrt{1+(V/v_0)^2}} & 0 & 0 \\ \frac{-(V/v_0)}{\sqrt{1+(V/v_0)^2}} & \frac{1}{\sqrt{1+(V/v_0)^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Invariance of Light Speed</u>

- 4-D rotation matrix – understood in 1800's
  - But no <u>physical</u> explanation of the arbitrary velocity v_0

- Electromagnetic Theory – understood in late 1800's
  - Described EM waves with speed c, but no physical medium
  - Michelson/Morley experiment showed there is no "ether"

- Einstein – If EM waves have no <u>material</u> medium:
  - Then there is no way to define a "special" reference frame
  - Thus, the wave speed must be c in every reference frame
  - This is <u>very</u> different behavior from a sound wave! (Doppler)
  - 4-D rotation must leave EM wave speed invariant
  - Could this be related to the "universal" velocity v_0?

Lorentz Transformation

- Applying the 4-D rotation matrix to a light wave:

$$\begin{pmatrix} v_0\, t' \\ c\, t' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+(V/v_0)^2}} & \frac{(V/v_0)}{\sqrt{1+(V/v_0)^2}} & 0 & 0 \\ \frac{-(V/v_0)}{\sqrt{1+(V/v_0)^2}} & \frac{1}{\sqrt{1+(V/v_0)^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_0\, t \\ c\, t \\ 0 \\ 0 \end{pmatrix}$$

Solve for v_0 in terms of c:

$$v_0^2 = -c^2$$

$$\boxed{v_0 = i\,c}$$

- Real numbers – not "flexible" enough for invariance of:
  - (length of 4-vectors) and (EM wave speed c)
  - Measurable 3-D quantities (distance, speed, etc.) must be real

Define:

$$\beta \equiv \frac{V}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

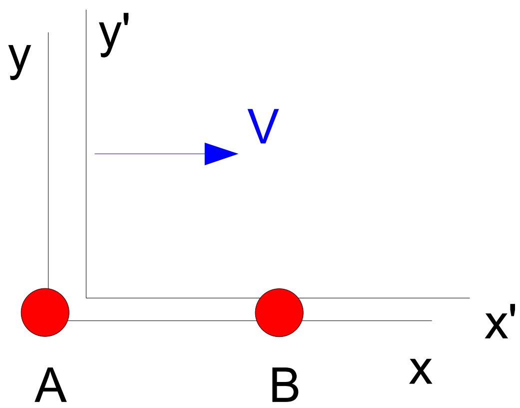
$$\begin{pmatrix} i\,c\,t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -i\,\gamma\,\beta & 0 & 0 \\ i\,\gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i\,c\,t \\ x \\ y \\ z \end{pmatrix}$$

Lorentz Transformation

Lorentz Transformation Examples

- Show that $L^{-1} L = L L^{-1} = I$
  - Where L^{-1} is the inverse Lorentz Transformation

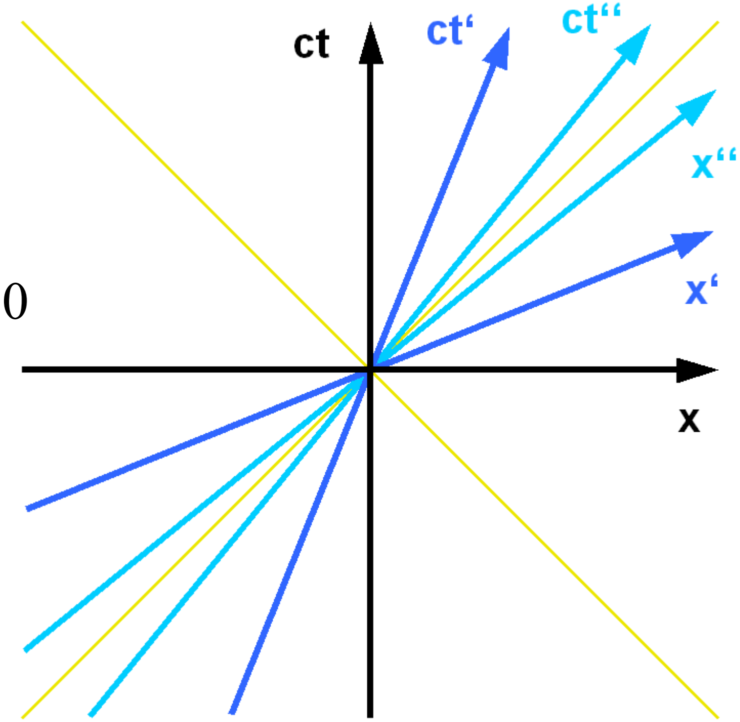
- Consider a previous example:



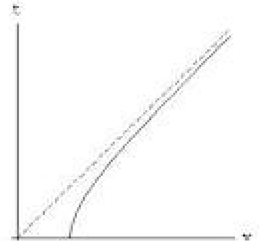
  - Event B: spatial origin of S' is at x=D at time t=T
  - Calculate position 4-vector of each event in each frame
  - Show that L preserves "length" of displacement 4-vector

Spacetime Diagrams

- Visualization tool for geometry in (x,ct) plane
  - Convention: x-axis horizontal; ct-axis vertical
  - x' axis → ct'=0 → $\gamma\left(c\,t - \beta\,x\right) = 0$
  - ct' axis → x'=0 → $\gamma\left(-\beta\,c\,t + x\right) = 0$

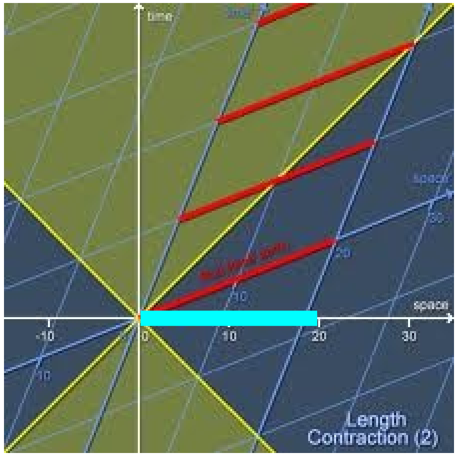


- Scale of x' and ct' axes
  - "Stretched" compared to x / ct axes
- Motion path in spacetime diagram: "worldline"
  - Series of events "occupied" by particle / observer
  - Light waves have slope of 45° (speed = c)



Length Contraction

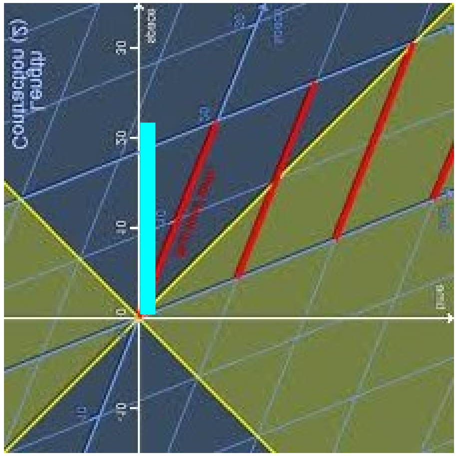
- Consider an object of “rest length” L_0
  - Which moves through the lab frame at speed V



- To measure “length” of object in any frame:
  - Need coordinates of 1) front / 2) back of object at same time
  - In other frames, these 2 events will not be simultaneous

- Example: at t'=0 – back end at x'=0, front end at $x'=L_0$
  - Calculate coordinates of front end at t=0
  - Length of object in lab frame appears shorter than L_0
  - Moving objects undergo “length contraction”

Time Dilation



- Consider an object with a "built-in clock"
  - Example: particle decay lifetime, human cells
- Object moves in lab frame at speed V
  - "Built-in clock" runs at an invariant rate called "proper time" (τ)
  - Clock in lab frame runs at a different rate – "time dilation"
- Example: π meson – lifetime at rest = 2.6 x 10^{-8} sec
  - Moving in lab at speed V = 0.6c → calculate lifetime in lab
- Twin paradox – one twin on Earth, other at speed V
  - If Earth twin ages 20 years, how much does moving twin age?
  - How does situation look from "moving" twin's rest frame?

Relativistic Doppler Effect

- Light waves → no medium → no "preferred" frame
  - Doppler effect used for sound waves can't apply (no v_L or v_S)
  - Only relative velocity of source and observer matters

- Consider a light source approaching at speed V
  - Frequency f_S (period T_S) – measured in source's frame
  - 1) detected λ is shortened – detector sees higher frequency:

$$\lambda_D = c\,T_D - V\,T_D = (c - V)\,T_D \qquad f_D = \frac{c}{\lambda_D} = \frac{c}{(c - V)\,T_D}$$

  - 2) time dilation – detector sees lower frequency $T_D = \gamma\,T_S$
  - Overall effect: $f_D = \sqrt{\dfrac{1+\beta}{1-\beta}}\,f_S$ $\qquad f_D = \dfrac{f_S}{\gamma\,(1 - \beta\,\cos\theta)}$

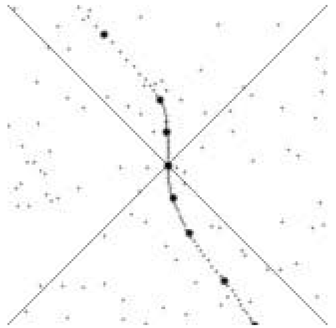
"Line of sight"

Doppler effect for general relative motion

4-Velocity

- Worldlines in 4-D spacetime → parameter τ: $\tilde{X}(\tau) = \begin{pmatrix} X_0(\tau) \\ X_1(\tau) \\ X_2(\tau) \\ X_3(\tau) \end{pmatrix}$
  - τ = "proper time" for the worldline – invariant

- How to define a velocity 4-vector?
  - Requirement: 4-D length must be invariant



- Examples:
  - Show that $\frac{\partial \tilde{X}}{\partial t}$ does not have invariant length
  - Show that $\frac{\partial \tilde{X}}{\partial \tau} = \gamma \frac{\partial \tilde{X}}{\partial t}$ does have invariant length
  - "4-Velocity" along worldline: $\tilde{U} \equiv \gamma_u \begin{pmatrix} i\,c \\ u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} U_0(\tau) \\ U_1(\tau) \\ U_2(\tau) \\ U_3(\tau) \end{pmatrix}$

At high speed:
$U_1 \to \infty$
$u_x \to c$

Examples

- Rocket moves relative to lab at speed V
  - Clocks are synchronized in usual way
  - i.e. (ct,x)=(0,0) and (ct',x')=(0,0) are same event
  - At t=T, lab frame emits light wave
  - Using mirrors, lab and rocket reflect light back and forth
  - 1) Draw a spacetime diagram for this situation
  - 2) Calculate times at which reflections occur (in each frame)

- Two rockets moving relative to lab frame:
  - Rocket A: moves at speed 0.8c in the +x direction
  - Rocket B: moves at speed 0.6c in the –x direction
  - Calculate speed of A, as measured by B

4-Acceleration

- Definition of 4-acceleration:

$$\tilde{A} \equiv \frac{d\,\tilde{U}}{d\,\tau} = \frac{d}{d\,\tau}\left(\gamma_u \begin{pmatrix} i\,c \\ u_x \\ u_y \\ u_z \end{pmatrix}\right)$$

  - In general, very complicated due to chain rule
  - Important "special case": v and a both along x-direction

$$\tilde{A} = \frac{d}{d\,\tau}\left(\gamma_u \begin{pmatrix} i\,c \\ u \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} i\,c\,\dfrac{d\,\gamma_u}{d\,\tau} \\ \dfrac{d\,\gamma_u}{d\,\tau}u + \gamma_u \dfrac{d\,u}{d\,\tau} \\ 0 \\ 0 \end{pmatrix} = \frac{d\,u}{d\,\tau}\gamma_u^3 \begin{pmatrix} i\,u/c \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Note: 4-acceleration reduces to 3-acceleration in "instantaneous rest frame" only

- Newton → acceleration same in every frame
  - Einstein → If 3-D acceleration is non-zero in one frame...
  - It is non-zero in every frame (but not invariant!)

4-Momentum

- Mass of a particle – invariant scalar quantity
  - 4-Velocity – valid 4-vector with invariant length
  - Thus, $\tilde{P} = m\,\tilde{U} = \begin{pmatrix} i\,\gamma\,m\,c \\ \gamma\,m\,u_x \\ \gamma\,m\,u_y \\ \gamma\,m\,u_z \end{pmatrix}$ is a 4-vector with invariant length
  - Called the "4-momentum" (a conserved quantity)
- At low speed: $(p_1, p_2, p_3) \rightarrow$ 3-momentum
  - What is the physical significance of the "time component"?

Relativistic Energy

- Time component of 4-momentum: $\tilde{P}_0 = i\,\gamma\,m\,c = i\,\dfrac{m\,c}{\sqrt{1-\dfrac{u^2}{c^2}}}$
  - To find Newtonian analog – examine low-speed limit:

$$\tilde{P}_0 = i\,m\,c\left[1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots\right] = i\left[m\,c + \frac{\frac{1}{2}\,m\,u^2}{c} + \ldots\right]$$

- 2nd term is Newtonian KE (divided by c)
  - 4-momentum contains both total energy and momentum
  - What does 1st term represent?
  - Einstein: mass itself is a form of "potential energy" $E = \gamma\,m\,c^2$
  - Known as particle's "rest energy" or "mass energy" $E_{rest} = m\,c^2$
  - With right conditions – can be converted to KE $KE = (\gamma - 1)\,m\,c^2$
  - Examples: nuclear reactions, matter/antimatter

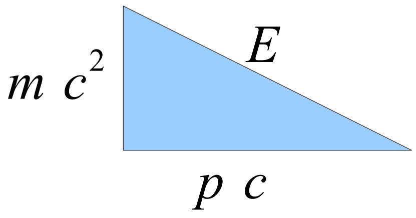
Energy-Momentum Relation

- 4-momentum of a particle: $\tilde{P} = \begin{pmatrix} \frac{iE}{c} \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$ $E = \gamma m c^2$ $\vec{p} = \gamma m \vec{u}$

- Invariant magnitude: $|\tilde{P}|^2 = p^2 - \frac{E^2}{c^2} = -m^2 c^2$

- Solving for the energy: $E^2 = p^2 c^2 + m^2 c^4$

- Can be viewed as a "Pythagorean theorem" for energy
  - Factors heavily in relativistic quantum mechanics



Photons

- Einstein proposed that light exists in discrete "packets"
  - Which move at the (invariant) speed of light
  - Photon concept also solved issues with other areas of physics
  - "Photoelectric effect" – experimental proof (Einstein Nobel Prize)

- Photons move at speed c: $\gamma \rightarrow \infty$
  - Only way for energy to be finite: $m \rightarrow 0$
  - Photons are pure KE: $E = pc$

Quantum Mechanics:

$$E_{photon} = hf$$

(f = frequency of light)

- Example: cart of length L and mass m_0 at rest
  - Photon with energy E is emitted from back of cart to front
  - Cart slides backward distance D due to momentum of photon
  - What mass m (moved from back to front) would yield same D?

Reactions

- Particle collisions / “explosions” (e.g. nuclear decay)
  - Classified according to energy in eV (1 eV = 1.6 x 10^{-19} J)

- Chemical reactions – nuclei and electrons stay same
  - “Binding energy” changes – exothermic or endothermic
  - Change in mass is miniscule but detectable (order of 1-10 eV)

- Nuclear reactions – nuclei “swap” protons / neutrons
  - Change in binding energy on the order of 1-10 MeV

- Particle reactions and matter / antimatter reactions
  - Particles change into different particles (e.g. $n \rightarrow p + e^{-} + v_e$)
  - Energy released on the order of 1-10,000 MeV

Center of Momentum Frame

- Newtonian system – CM gives "preferred" frame
  - Total momentum = 0 and all energy is "internal" to system
- Relativistic system – CM yields non-zero momentum
  - More useful: "center of momentum" frame
- Example:

  - Newtonian CM frame moves to right at speed c/4
  - Calculate V for frame in which total momentum is zero
  - If particles collide and "stick" – what is mass of final particle?
- Collisions often simplest in center of momentum frame
  - Example: Calculate final particle mass in both frames

Example

- Compton scattering
  - Photon with energy E_0 has glancing collision with electron
  - Transfers some energy to electron
  - Final photon moves at angle θ from initial line of motion
  - Calculate energy of final photon

- In previous example:
  - Calculate speed of center of momentum frame if $E_0 = m_e c^2$
  - Using the Lorentz transformation:
  - Find the final 4-momentum of photon in CM frame
  - What is the angle θ' of final photon in CM frame?

Accelerating Reference Frames

- Consider an rocket with a "light clock"
  - Photons are emitted at back of rocket, detected at front
- When rocket is accelerating:
  - Speed changes in time between photon emission / detection
  - Relative speed between emitter / detector → Doppler shift
- Acceleration causes Newtonian "fictitious" force field
  - Indistinguishable from effects of uniform gravitational field
  - "Equivalence principle" – gravity / "fictitious" force identical
  - Confirmed by measurement of "gravitational redshift" of light
  - Einstein's General Relativity – physics in accelerating frames
  - 4-D spacetime in these frames is said to be "curved"