Complex Numbers

Complex Algebra

The set of complex numbers can be defined as the set of pairs of real numbers, $\{(x, y)\}$, with two operations: (i) addition,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2) \ , \tag{1}$$

and (ii) *complex multiplication*,

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1\, x_2 - y_1\, y_2,\ x_1\, y_2 + x_2\, y_1) \ . \tag{2}$$

This two operations define *complex algebra.*

⋄ With the rules(1)-(2), complex numbers include the real numbers as a subset $\{(x, 0)\}$ with usual real number algebra. This suggests short-hand notation $(x, 0) \equiv x$; in particular: $(1, 0) \equiv 1$.

⋄ Complex algebra features commutativity, distributivity and associativity.

The two numbers, $1 = (1, 0)$ and $i = (0, 1)$ play a special role. Each complex number can be represented in a unique way as [we start using the notation $(x, 0) \equiv x$]

$$(x, y) = x + iy \ . \tag{3}$$

⋄ Terminology: The number i is called imaginary unity. For the complex number $z = (x, y)$, the real umbers x and y are called real and imaginary parts, respectively; corresponding notation is: $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$.

The following remarkable property of the number i,

$$i^2 \equiv i \cdot i = -1 \ , \tag{4}$$

(in combination with commutativity, distributivity and associativity) renders representation (3) most convenient for practical algebraic manipulations with complex numbers.—One treats x, y, and i the same way as the real numbers.

Another useful parametrization of complex numbers follows from the geometrical interpretation of the complex number $z = (x, y)$ as a point in a 2D plane, referred to in this context as *complex plane.* Introducing polar coordinates, the radius $r = \sqrt{x^2 + y^2}$ and the angle $\theta = \tan^{-1}(y/x)$, one gets

$$x + iy = r(\cos\theta + i\sin\theta)\ . \tag{5}$$

⋄ Terminology and notation: Radius r is called *modulus* (and also *magnitude*) of the complex number, $r = |z|$. The angle θ is called *phase* (and also *argument*) of the complex number, $\theta = \arg(z)$. Note an ambiguity in the definition of the phase of a complex number. It is defined up to an additive multiple of 2π.

Since the modulus of a complex number is nothing else than the magnitude of corresponding vector, the standard vector inequalities are applicable:

$$\big|\,|z_1| - |z_2|\,\big| \ \le\ |z_1 + z_2| \ \le\ |z_1| + |z_2|\ . \tag{6}$$

Parametrization in terms of modulus and phase is convenient for multiplication, because if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]\ . \tag{7}$$

Polar parametrization is also convenient for the division which is considered below.

Subtraction and division of complex numbers are defined as the operations opposite to addition and multiplication, respectively. Division of complex numbers can be actually reduced to multiplication. But first we need to introduce one more important operation, *complex conjugation.* For each complex number $z = x + iy$ we define its complex conjugate as

$$z^* = x - iy \tag{8}$$

and note that

$$zz^* = |z|^2 \tag{9}$$

is a real number. Then for any two complex numbers z_1 and z_2 the operation of division can be written as

$$\frac{z_1}{z_2} = |z_2|^{-2} z_1 z_2^*\ . \tag{10}$$

The validity of this relation is checked by multiplying the right-hand side by z_2. In modulus-phase parametrization, Eq. (10) reads

$$z_1/z_2 \ =\ (r_1/r_2)\,[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]\ . \tag{11}$$

For the complex fractions, the following statement is true (and easy to prove):

$$\frac{z_1}{z_2} = \frac{z_1\, z_0}{z_2\, z_0} \, , \tag{12}$$

where z_1, z_2, and z_0 are arbitrary complex numbers, provided $z_2 \neq 0$ and $z_0 \neq 0$. Note that if we take $z_0 = z_2^*$, we reproduce Eq. (10).

Since the operation of complex conjugation plays a crucial part in the theory of complex numbers, it is important to know how to complex conjugate algebraic expressions. This is done by using two simple relations

$$(z_1 + z_2)^* = z_1^* + z_2^* \, , \tag{13}$$

$$(z_1 \cdot z_2)^* = z_1^* \cdot z_2^* \, . \tag{14}$$

The other useful relations are

$$z + z^* = 2\,\mathrm{Re}\, z \, , \tag{15}$$

$$z - z^* = 2\, i \mathrm{Im}\, z \, . \tag{16}$$

Finally, it is useful to keep in mind that $z = z^*$ iff z is real, while $z = -z^*$ iff z is imaginary ('iff' means 'if and only if').

Problem 1. Establish relations between complex multiplication and inner/outer vector product: Treating the two pairs, (x_1, y_1), (x_2, y_2), as both two vectors in the xy plane, $(x_1, y_1) = \vec{a}$, $(x_2, y_2) = \vec{b}$, and two complex numbers, $(x_1, y_1) = a$, $(x_2, y_2) = b$, make sure that

$$\vec{a} \cdot \vec{b} = (1/2)(a\, b^* + b\, a^*) \, , \tag{17}$$

$$\vec{a} \times \vec{b} = (i/2)(a\, b^* - b\, a^*)\, \hat{\mathbf{z}} \, , \tag{18}$$

where $\hat{\mathbf{z}}$ is the unit vector along the z-direction.

Functions of a Complex Variable

A complex function $w = u + iv$ of a complex variable $z = x + iy$ is introduced as a complex-valued function of two real variables, x and y:

$$w(z) = u(x, y) + iv(x, y) \, . \tag{19}$$

Hence, to specify a complex function it is enough to specify two real functions: $u(x, y)$ and $v(x, y)$.

Partial derivatives are defined as

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\,, \qquad\qquad \frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\,. \tag{20}$$

We now formally define partials $\partial/\partial z$ and $\partial/\partial z^*$ as

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\,\frac{\partial w}{\partial y}\right)\,, \qquad \frac{\partial w}{\partial z^*} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\,\frac{\partial w}{\partial y}\right)\,. \tag{21}$$

The idea behind definitions (21) is in the following observation. Suppose $w(z)$ is specified not in terms of x and y, but in the form of some finite algebraic expression or infinite series, $w(z, z^*)$, involving z and z^*. In this expression one can formally replace complex numbers z and z^* with two *independent* real variables: $z \to a$, $z^* \to b$ and arrive at the function $w(a, b)$. It is easy to see then from (20)-(21) that $\partial w(z, z^*)/\partial z$ is equal to $\partial w(a, b)/\partial a$, $\;a \to z$, $b \to z^*$, and $\partial w(z, z^*)/\partial z^*$ is equal to $\partial w(a, b)/\partial b$, $\;a \to z$, $b \to z^*$. That is with respect to the operations $\partial/\partial z$ and $\partial/\partial z^*$ the variables z and z^* behave as independent real variables. This essentially simplifies calculation of partials. [For example, if $w(z, z^*) = zz^*$, then $\partial w/\partial z = z^*$ and $\partial w/\partial z^* = z$.]

Problem 2. Prove the above-mentioned general property of the operations $\partial/\partial z$ and $\partial/\partial z^*$. *Hint.* Since the formal rules of complex and real algebras are the same, the standard differentiating rules are applicable to complex-valued functions when differentiated with respect to x and y.

Consider an infinitesimal variation, δw, of the function $w(z, z^*)$ corresponding to $z \to z + \delta z$, where $\delta z = \delta x + i\delta y$ (and implying $z^* \to z^* + \delta z^*$, $\delta z^* = \delta x - i\delta y$). As can be readily checked with the definitions (21),

$$\delta w = \delta z\,\frac{\partial w}{\partial z} + \delta z^*\,\frac{\partial w}{\partial z^*}\,. \tag{22}$$

Problem 3. Check Eq. (22).

Note that while δz and δz^* essentially depend on each other, the expression (22) formally looks like they were independent variables.

How do we construct complex functions? The simplest way is to take a real expression involving four arithmetic operations with one (or two) real numbers a (and b) and replace in it a with a complex variable z (and b with z^*). A more powerful way is to use a power series.

A very important sub-set of complex functions is formed by functions that depend only on z, but not on z^*—in the sense that corresponding real arithmetic expression (or power series) involves only one variable, a, which is then replaced with z. Clearly, for all such functions the operation $\partial/\partial z^*$ yields zero.
Examples:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \ , \tag{23}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \ , \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \ , \tag{24}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \ , \qquad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \ , \tag{25}$$

All the series in (23)-(25) are convergent and the functions are well defined for any z, coinciding with corresponding real functions at real z. *Very important:* The complex functions defined this way feature *all* the functional and differential relations characteristic of corresponding real functions, because (i) these relations are captured algebraically by the power series and (ii) the real and complex algebras coincide. For example,

$$e^{z_1+z_2} \ = \ e^{z_1} e^{z_2} \ , \tag{26}$$

$$(e^z)^\alpha \ = \ e^{\alpha z} \tag{27}$$

(strictly speaking, at this point we can discuss only integer α's, since we have not defined yet the notion of a real-valued power function)

$$\frac{\partial e^z}{\partial z} \ = \ e^z \ , \tag{28}$$

$$\frac{\partial \sin z}{\partial z} \ = \ \cos z \ , \tag{29}$$

$$\sin(z_1 + z_2) \ = \ \sin z_1 \, \cos z_2 + \cos z_1 \, \sin z_2 \ , \tag{30}$$

etc. Quite amazingly, *new* functional relations arise.
Examples:

$$e^{iz} = \cos z + i \sin z \ , \tag{31}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \cosh{(iz)} \ , \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} = -i \sinh{(iz)} \ , \tag{32}$$

$$\cosh z = \cos(iz) , \qquad \sinh z = -i \sin(iz) . \tag{33}$$

Problem 4. Prove Eqs. (31)-(33) by direct comparison of power series.

With the relation (31), the polar representation of a complex number, Eq. (5), can be written as

$$z = re^{i\theta} , \tag{34}$$

after which relations (7) and (11) become most transparent.

Now we are in a position to define any real power of a complex number $z = re^{i\theta}$. By definition,

$$z^{\alpha} = r^{\alpha} e^{i\alpha\theta} . \tag{35}$$

Note that if α is not integer, then z^{α} is not single valued, because it depends on the choice of θ, and θ is defined only up to a multiple of 2π. For example, if we parameterize -1 as

$$-1 = e^{i\pi} , \tag{36}$$

we get

$$(-1)^{1/2} = e^{i\pi/2} = i . \tag{37}$$

But if we take the parametrization

$$-1 = e^{-i\pi} , \tag{38}$$

we arrive at

$$(-1)^{1/2} = e^{-i\pi/2} = -i . \tag{39}$$

And both results are equally correct. They simply mean that the equation

$$z^2 = -1 \tag{40}$$

has two different roots: $z = i$ and $z = -i$.

Problem 5. Find all complex roots z of the equation

$$z^n = a , \tag{41}$$

where $a = |a|\mathrm{e}^{i\theta}$ is some given complex number, and n is some given natural number.

Solving Harmonic Oscillator Problem with Complex Numbers

The problem of harmonic oscillator plays a very important role in physics, since it describes small oscillations, characteristic of any stable mechanical system. In the absence of dissipation and driving forces, the equation of motion of harmonic oscillator reads

$$\ddot{\eta} + \omega_0^2 \eta = 0 , \tag{42}$$

where $\eta \equiv \eta(t)$ is a certain (generalized) coordinate, as a function of time; ω_0 is the angular frequency. Mathematically, Eq. (42) is a linear homogeneous second-order ordinary differential equation with time-independent coefficients. [Homogeneous means zero right-hand side, second-order means no derivatives higher than second derivative, ordinary means no partial derivatives. Further in this course, we will see that complex numbers yield a perfect universal tool for solving linear homogeneous ordinary differential equations (and systems) with constant coefficients. In this chapter, we confine ourselves with only one particular example: harmonic oscillator.]

The trick is to solve for a complex-valued function $w(t)$ satisfying Eq. (42), and then take its real and imaginary parts. The two have to be the real solutions of Eq. (42). Indeed,

$$\ddot{w} + \omega_0^2 w = 0 \tag{43}$$

implies

$$\ddot{w}_1 + i\ddot{w}_2 + \omega_0^2 w_1 + i\omega_0^2 w_2 = 0 , \tag{44}$$

where

$$w_1 = \mathrm{Re}\, w , \qquad w_2 = \mathrm{Im}\, w . \tag{45}$$

But Eq. (43) is equivalent to two real-valued equations:

$$\ddot{w}_1 + \omega_0^2 w_1 = 0 , \qquad \ddot{w}_2 + \omega_0^2 w_2 = 0 , \tag{46}$$

each of which is nothing but Eq. (42). Of course, the crucial question is: Why is the complex-valued equation simpler than its real-valued counterpart? And the answer is that with complex-valued equation we can use the exponential substitution

$$w = A\, \mathrm{e}^{\lambda t} , \tag{47}$$

where A and λ are some complex numbers. This substitution reduces Eq. (43) to a purely *algebraic* equation for λ

$$\lambda^2 + \omega_0^2 = 0 . \tag{48}$$

The constant A drops out because of the linearity of the equation. This means that any A is allowed, provided λ satisfies Eq. (48).

With real numbers, Eq. (48) makes no sense, while with complex numbers we readily get two solutions:

$$\lambda \;=\; \pm i\omega_0 \;, \tag{49}$$

where without loss of generality we can assume that $\omega_0 \geq 0$. Because of the linearity of Eq. (43), its general solution is just a sum of special solutions we just found:

$$w(t) \;=\; A_1\,\mathrm{e}^{i\omega_0 t} \;+\; A_2\,\mathrm{e}^{-i\omega_0 t} \;, \tag{50}$$

where A_1 and A_2 are any complex numbers. To get back to the real-valued solutions, it is convenient to represent A_1 and A_2 in the polar form:

$$A_1 \;=\; a_1\,\mathrm{e}^{i\varphi_1} \;, \qquad\qquad A_2 \;=\; a_2\,\mathrm{e}^{i\varphi_2} \;, \tag{51}$$

where $a_1 = |A_1|$ and $a_2 = |A_2|$. This yields

$$w(t) \;=\; a_1\,\mathrm{e}^{i(\omega_0 t+\varphi_1)} \;+\; a_2\,\mathrm{e}^{i(\varphi_2-\omega_0 t)} \;, \tag{52}$$

and taking real and imaginary parts of this solution, we find

$$\eta_1(t) \;=\; a_1\cos(\omega_0 t+\varphi_1) \;+\; a_2\cos(\varphi_2-\omega_0 t) \;, \tag{53}$$

$$\eta_2(t) \;=\; a_1\sin(\omega_0 t+\varphi_1) \;+\; a_2\sin(\varphi_2-\omega_0 t) \;. \tag{54}$$

In fact, the two solutions, η_1 and η_2 are identical to each other, since sines can be easily transformed into cosines by using the freedom of choosing φ_1 and φ_2. Moreover, each of the two real-valued solution is overdefined in terms of the number of free constants. For a second-order equation, there should be only two independent free constants. Still, this overdefined form is quite convenient since it allows one to get rid of any two constants by setting them equal to some special values. For example, setting $\varphi_1 = \pi/2$ or $\varphi_2 = \pi/2$ transforms corresponding cosine into sine, and vice versa. This way we get two standard forms of representing the solution:

$$\eta(t) \;=\; a\cos(\omega_0 t-\varphi_0) \;, \tag{55}$$

and

$$\eta(t) \;=\; c_1\cos(\omega_0 t) \;+\; c_2\sin(\omega_0 t) \;. \tag{56}$$

Now consider a damped harmonic oscillator:

$$\ddot{\eta} + \gamma\dot{\eta} + \omega_0^2\,\eta = 0\,, \tag{57}$$

where $\gamma > 0$ is the damping coefficient. The idea of solving this equation is the same. We find the solution of a complex equation

$$\ddot{w} + \gamma\dot{w} + \omega_0^2\,w = 0\,, \tag{58}$$

and then take either real or imaginary part of w.

Substitution (47) reduces equation (58) to the algebraic equation

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0\,. \tag{59}$$

Depending on the strength of the damping term, the two roots of the equation (59) are either real,

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm \sqrt{(\gamma/2)^2 - \omega_0^2} \qquad (\gamma > 2\omega_0)\,, \tag{60}$$

or complex,

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - (\gamma/2)^2} \qquad (\gamma < 2\omega_0)\,. \tag{61}$$

The real roots that take place at $\gamma > 2\omega_0$ correspond to so-called overdamped regime. Actually, in this case we do not need to use the complex substitution. The exponentials turn out to be real, and we immediately write down the solution in the form of two decaying terms

$$\eta(t) = c_1\,\mathrm{e}^{-(\gamma/2+\sqrt{(\gamma/2)^2-\omega_0^2})t} + c_2\,\mathrm{e}^{-(\gamma/2-\sqrt{(\gamma/2)^2-\omega_0^2})t}\,, \tag{62}$$

where c_1 and c_2 are real coefficients.

The case $\gamma < 2\omega_0$ is the so-called underdamped regime. Here we deal with an essentially complex solution

$$w(t) = A_1\,\mathrm{e}^{\lambda_1 t} + A_2\,\mathrm{e}^{\lambda_2 t}\,, \tag{63}$$

which we parameterize as

$$w(t) = \mathrm{e}^{-\gamma t/2}\,[a_1\,\mathrm{e}^{i(\tilde{\omega}t+\varphi_1)} + a_2\,\mathrm{e}^{i(\varphi_2-\tilde{\omega}t)}]\,, \tag{64}$$

where

$$\tilde{\omega} = \sqrt{\omega_0^2 - (\gamma/2)^2}\,, \tag{65}$$

and $a_1 = |A_1|$, $a_2 = |A_2|$; φ_1 and φ_2 are the phases of A_1 and A_2, respectively.

Taking real and imaginary parts of the complex solution, we get two real solutions

$$\eta_1(t) \;=\; \mathrm{e}^{-\gamma t/2}\,[a_1 \cos(\tilde{\omega} t + \varphi_1) \,+\, a_2 \cos(\varphi_2 - \tilde{\omega} t)] \;, \tag{66}$$

$$\eta_2(t) \;=\; \mathrm{e}^{-\gamma t/2}\,[a_1 \sin(\tilde{\omega} t + \varphi_1) \,+\, a_2 \sin(\varphi_2 - \tilde{\omega} t)] \;. \tag{67}$$

The two are identical to each other because of too many free constants. Once again we can use this freedom to select this or that particular form of parametrization. The two frequently used forms are

$$\eta(t) \;=\; a\mathrm{e}^{-\gamma t/2} \cos(\tilde{\omega} t - \varphi_0) \;, \tag{68}$$

and

$$\eta(t) \;=\; \mathrm{e}^{-\gamma t/2}\,[c_1 \cos(\tilde{\omega} t) \,+\, c_2 \sin(\tilde{\omega} t)] \;. \tag{69}$$

We see that the underdamped regime is reminiscent of the regime without damping. The new qualitative feature is vanishing of the amplitude of oscillations with time. Interestingly enough, the damping term changes the frequency of oscillations.

Finally, we consider the so-called driven motion of harmonic oscillator, when the is an external periodic force acting on the system. The equation of motion reads

$$\ddot{\eta} \,+\, \gamma \dot{\eta} \,+\, \omega_0^2\, \eta \;=\; f(t) \;, \tag{70}$$

with

$$f(t) \;=\; a \cos \omega t \;. \tag{71}$$

A nice feature of this equation is that its left-hand side is linear. This allows us to look for a general solution in the form of a sum of a general solution of corresponding homogeneous equation (that is the equation with $f \equiv 0$), and *any* special solution of the given equation. Since we have already found the general solution of the homogeneous equation, we just need to find some particular solution of the equation (70). We start with noting that

$$f(t) \;=\; \mathrm{Re}\, a\, \mathrm{e}^{i\omega t} \;. \tag{72}$$

Hence, if we find any particular solution to the complex equation

$$\ddot{w} \,+\, \gamma \dot{w} \,+\, \omega_0^2\, w \;=\; a\, \mathrm{e}^{i\omega t} \tag{73}$$

and then take its real part, we will get what we need: a particular solution to the equation (70). The form of the equation suggests the exponential substitution

$$w \;=\; B\,\mathrm{e}^{i\omega t} \,. \tag{74}$$

Note that now the angular frequency ω is not a free parameter. It comes from the frequency of the external force. The only free parameter is the amplitude B.

The substitution reduces Eq. (73) to algebraic equation for B:

$$B\,(\omega_0^2 \,-\, \omega^2 \,+\, i\gamma\omega) \;=\; a \,. \tag{75}$$

We thus have

$$B \;=\; \frac{a}{\omega_0^2 \,-\, \omega^2 \,+\, i\gamma\omega} \;=\; \frac{a\,\mathrm{e}^{-i\varphi_0}}{\sqrt{(\omega_0^2 \,-\, \omega^2)^2 \,+\, \gamma^2\,\omega^2}} \,, \tag{76}$$

where

$$\varphi_0 \;=\; \tan^{-1}\frac{\gamma\omega}{\omega_0^2 \,-\, \omega^2} \,. \tag{77}$$

[Polar representation for the complex number B proves most convenient.] Finally, taking the real part of the complex solution w we get our real solution

$$\eta(t) \;=\; \frac{a}{\sqrt{(\omega_0^2 \,-\, \omega^2)^2 \,+\, \gamma^2\,\omega^2}}\,\cos(\omega t - \varphi_0) \,. \tag{78}$$

Let us discuss the physics behind the solution (78). First, it is worth noting that as t goes to infinity, only this particular solution survives, since the general equation of the homogeneous equation (exponentially) decays with time. Secondly, it is important that the system oscillates with the frequency of the driving force, its own frequency being not relevant at large t. It is also important that there is a phase shift φ_0 between oscillations of the force and the response of the system.

If $\gamma \ll \omega_0$, then the amplitude $|B|$ has a peaked shape with the characteristic width of the order of γ. The resonance takes place at the frequency $\omega_R \approx \omega_0$ and the amplitude at resonance is perfectly approximated by the value of $|B|$ at $\omega \,=\, \omega_0$:

$$|B|_R \;\approx\; \frac{a}{\omega_0\gamma} \qquad\qquad (\gamma \;\ll\; \omega_0) \,. \tag{79}$$

With increasing γ the peak becomes broader and its amplitude decreases. At $\gamma \sim \omega_0$ the position of the peak significantly shifts from ω_0 to lower frequencies, as is seen from the exact formula for the resonance frequency (the frequency corresponding to the maximal $|B|$):

$$\omega_R \;=\; \sqrt{\omega_0^2 - \gamma^2/2}\;. \tag{80}$$

From this relation we also see that the position of the peak reaches zero frequency at $\gamma \;=\; \sqrt{2}\,\omega_0$.

Problem 6. Derive Eq. (80) and find the exact relation for $|B|_R$. Show that at $\gamma \;>\; \sqrt{2}\,\omega_0$ the maximal $|B|$ corresponds to $\omega = 0$, and $|B|$ monotonically decreases with increasing ω.

Charged Particle in Homogeneous Magnetic Field

Consider a particle of mass m and electric charge q, moving in a homogeneous magnetic field $\mathbf{B}$. The equation of motion reads

$$m\mathbf{a} \;=\; \mathbf{F}\;, \tag{81}$$

where $\mathbf{a} \;=\; \dot{\mathbf{v}} \;=\; \ddot{\mathbf{r}}$ is the acceleration ($\mathbf{r}$ is the radius vector and $\mathbf{v}$ is the velocity), and

$$\mathbf{F} \;=\; q\,\mathbf{v}\times\mathbf{B}\;. \tag{82}$$

Selecting z-axis in the direction of $\mathbf{B}$, we have

$$\dot{\mathbf{v}} \;=\; \omega_0\mathbf{v}\times\hat{e}_z\;, \tag{83}$$

where

$$\omega_0 \;=\; \frac{qB}{m}\;. \tag{84}$$

Writing Eq. (83) in components we get

$$\dot{v}_z \;=\; 0\;, \tag{85}$$

$$\dot{v}_x \;=\; \omega_0\,v_y\;, \tag{86}$$

$$\dot{v}_y \;=\; -\omega_0\,v_x\;. \tag{87}$$

Our first observation is that the motion along the z-axis decouples from the xy-motion, and is actually trivial: The velocity v_z is constant, hence $z(t) = z_0 + v_z t$.

Now we can forget about the z-coordinate and concentrate on the xy-motion. Introducing the complex variable

$$w \; = \; v_x \, + \, i v_y \; , \tag{88}$$

we note that the two equations, (86) and (87), are actually equivalent to one very simple complex equation

$$i \, \dot{w} \; = \; \omega_0 \, w \; . \tag{89}$$

This equation is immediately solved by the exponential substitution

$$w \; = \; A \, \mathrm{e}^{\lambda t} \; , \tag{90}$$

leading to a trivial algebraic equation for λ:

$$i\lambda \; = \; \omega_0 \; . \tag{91}$$

Parameterizing $A = a\mathrm{e}^{i\varphi}$, we have

$$w \; = \; a\mathrm{e}^{i(\varphi - \omega_0 t)} \; , \tag{92}$$

and returning back to v_x and v_y, by taking real and imaginary parts, we find

$$v_x \; = \; a \cos(\omega_0 t - \varphi) \; , \tag{93}$$

$$v_y \; = \; -a \sin(\omega_0 t - \varphi) \; . \tag{94}$$

To find the coordinates we can simply integrate the expressions (93)-(94) with respect to time. Alternatively, we can integrate the complex solution (92). Introducing the complex variable

$$R \; = \; x \, + \, iy \; , \tag{95}$$

which encodes x and y coordinates in its real and imaginary parts, and noting that

$$\dot{R} \; = \; w \qquad \Rightarrow \qquad R \; = \; \int_0^t w(t')dt' \, + \, R_0 \; , \tag{96}$$

where $R_0 = R(t=0)$, we get

$$R(t) \; = \; a \int_0^t \mathrm{e}^{i(\varphi - \omega_0 t')} dt' \, + \, R_0 \; . \tag{97}$$

Problem 7. Find $R(t)$ by performing the integral (97). Note that the rules of integration of complex-valued functions with respect to real variables are absolutely the same as for corresponding real-valued functions. Compare the real and imaginary parts of your result for $R(t)$ with the results of direct integration of the expressions (93)-(94) with respect to t. Show that the trajectory of the particle in the xy plane is a circle.