Jacobians
(Change of Variables in a Multidimensional Integral)

Suppose we need to do some integral to evaluate some physical quantity of interest. For example, if we have a ball of radius R and mass density η, rotating about its axis with the constant angular velocity ω, we might be interested in finding the total kinetic energy associated with the rotation. The kinetic energy is the integral over all the infinitesimal elements of the ball:

$$E = \int dE \; . \tag{1}$$

Here

$$dE = \frac{[v(\mathbf{r})]^2 dm}{2} = \frac{\eta [v(\mathbf{r})]^2 dV}{2} \; , \tag{2}$$

where $dm = \eta dV$ is the mass of the infinitesimal element, dV is the volume of the infinitesimal element, and $v(\mathbf{r})$ is absolute value of the velocity of the infinitesimal element with the position $\mathbf{r}$. With the z-axis being the axis of the rotation, we have

$$v(\mathbf{r}) = \omega \sqrt{x^2 + y^2} \; , \tag{3}$$

and taking into account that $dV = dx\, dy\, dz$ we arrive at the following mathematical expression:

$$E = (\eta \omega^2/2) \int_{x^2+y^2+z^2 \le R^2} (x^2 + y^2)\, dx\, dy\, dz \; . \tag{4}$$

We see that the main difficulty with doing this integral comes from the domain of integration which does not have a simple form in the Cartesian coordinates. The spherical coordinates,

$$\begin{cases} x = r \sin\theta \cos\varphi \, , \\ y = r \sin\theta \sin\varphi \, , \\ z = r \cos\theta \, , \end{cases} \tag{5}$$

would be much more natural, since the domain of integration is defined for each variable independently of the others: $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$, $r \in [0, R]$. But to be able to do the integration in spherical coordinates we, generally speaking, need to know how to change the variables in the multi-dimensional integrals. The prescription is given by the following theorem. For any two sets of n variables, $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, and for an arbitrary function f, the following is true:

$$\int f\, dx_1 dx_2 \cdots dx_n = \int f\, |J^{(xy)}|\, dy_1 dy_2 \cdots dy_n \; , \tag{6}$$

where $J^{(xy)}$ is a determinant of a $n \times n$ matrix $A^{(xy)}$, the elements of which are defined as follows

$$A_{ij}^{(xy)} = \frac{\partial x_i}{\partial y_j} \; . \tag{7}$$

The determinant $J^{(xy)}$ is called *Jacobian*. In the r.h.s. of Eq. (6), both the function f and the Jacobian are supposed to be expressed as functions of $y_1, y_2, \ldots, y_n$. There is a more explicit symbol for Jacobians that is reminiscent of that for partial derivatives:

$$J^{(xy)} \equiv \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \; . \tag{8}$$

One of the advantages of this symbol is that it explicitly lists both sets of variables. Another advantage is that its formal structure of a fraction plays a mnemonic role for one very important property of Jacobians which we are going to establish. Suppose there is yet another set of n variables, $z = (z_1, z_2, \ldots, z_n)$. Then for the partial derivative of x_i with respect to z_j (all other z's being fixed) we can write the chain rule expression

$$\frac{\partial x_i}{\partial z_j} = \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k} \frac{\partial y_k}{\partial z_j} . \tag{9}$$

In terms of the above-introduced matrix of partial derivatives, relation (9) means

$$A^{(xz)} = A^{(xy)} A^{(yz)} . \tag{10}$$

Recalling that the determinant of a product of matrices equals to the product of determinants, we get

$$J^{(xz)} = J^{(xy)} J^{(yz)} , \tag{11}$$

or, using the symbol (8),

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(z_1, z_2, \ldots, z_n)} = \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(z_1, z_2, \ldots, z_n)} . \tag{12}$$

In particular, setting $z = x$ [and taking into account that $A^{(xx)} = I$ and thus $J^{(xx)} = \det I = 1$] we get

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \left[\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \right]^{-1} . \tag{13}$$

The general property of the determinants—the change of sign upon exchanging either two rows, or two columns implies that the sign of the Jacobian changes upon exchanging of the positions of two variables either in 'numerator', or 'denominator' of Eq. (8). [Clearly, this change of sign is not relevant to the problem of changing variables in integrals, where only the absolute values of Jacobians matter.]

Another—almost trivial, but quite important—property of Jacobians is that if both 'numerator' and 'denominator' contain one and the same variable, this variable can be eliminated, reducing thereby the rank of the matrix by one while preserving the value of the determinant.

Now we are in a position to see how the Jacobian technique works in most characteristic cases. For the spherical coordinates we have

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = r^2 \sin\theta , \tag{14}$$

which, in particular, allows us to readily solve the above-mentioned problem of kinetic energy of rotating ball. One practical advice, which proves to be useful for virtually any case of integration in spherical coordinates, is to employ the variable

$$\chi = \cos\theta \tag{15}$$

instead of θ. The point is that typically the dependence of the function f on θ is either absent, or comes in the form of (or can be reduced to) the dependence on $\cos\theta$, and then

$$\int_0^{\pi} f(\cos\theta)\, \sin\theta\, d\theta = \int_{-1}^{1} f(\chi)\, d\chi . \tag{16}$$

Problem 34. Derive Eq. (14) by explicitly evaluating the Jacobian as the determinant of 3×3 matrix. Find the kinetic energy of the rotating ball by doing the integral (4) in spherical coordinates.

Equation (12) allows one to significantly simplify the evaluation of Jacobians by splitting the change of variables into elementary pairwise steps. Let us illustrate this by going from Cartesian coordinates (x, y, z) to the spherical coordinates (r, θ, φ) in two steps: (i) going from Cartesian (x, y, z) to cylindrical coordinates (ρ, φ, z):

$$\begin{cases} x = \rho \cos\varphi \,, \\ y = \rho \sin\varphi \,, \end{cases} \tag{17}$$

and (ii) going from cylindrical (ρ, φ, z) to spherical coordinates (r, θ, φ):

$$\begin{cases} \rho = r \sin\theta \,, \\ z = r \cos\theta \,. \end{cases} \tag{18}$$

From the properties of the Jacobians we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \frac{\partial(x,y,z)}{\partial(\rho,\varphi,z)} \frac{\partial(\rho,\varphi,z)}{\partial(r,\theta,\varphi)} = -\frac{\partial(x,y,z)}{\partial(\rho,\varphi,z)} \frac{\partial(\rho,z,\varphi)}{\partial(r,\theta,\varphi)} = -\frac{\partial(x,y)}{\partial(\rho,\varphi)} \frac{\partial(\rho,z)}{\partial(r,\theta)} \,. \tag{19}$$

Clearly, the cylindrical coordinates (which are nothing but polar coordinates with respect to in the xy-plane, ρ being the polar radius) are important on their own, and corresponding Jacobian,

$$\frac{\partial(x,y)}{\partial(\rho,\varphi)} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho \,, \tag{20}$$

is thus useful per ce. It says that going from Cartesian coordinates to polar/cylindrical ones implies $dx\, dy \;\to\; \rho\, d\rho\, d\varphi$.

Now we calculate

$$\frac{\partial(\rho,z)}{\partial(r,\theta)} = \begin{vmatrix} \sin\theta & r\cos\theta \\ \cos\theta & -r\sin\theta \end{vmatrix} = r \,, \tag{21}$$

and readily conclude that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \rho r = r^2 \sin\theta \,. \tag{22}$$

Let us establish natural coordinates for working with a torus. To this end we note that if z is the symmetry axis of the torus, then one of the natural coordinates is the azimuthal angle $\varphi \in [0, 2\pi)$ (the same as in spherical and cylindrical coordinates). Given the fact that the cross-section of the torus by a half-plane $\varphi = \text{const}$ (see Fig. 1) is a circle of radius R_1 [with the center at the distance R_0 from the z-axis], it is natural to introduce the polar coordinates $r' \in [0, R_1]$, $\alpha \in [0, 2\pi)$ that conveniently characterize each point of the cross-section. The Jacobian of the transformation is easily found by

$$\frac{\partial(x,y,z)}{\partial(r',\alpha,\varphi)} = \frac{\partial(x,y,z)}{\partial(\rho,\varphi,z)} \frac{\partial(\rho,\varphi,z)}{\partial(r',\alpha,\varphi)} = -\frac{\partial(x,y)}{\partial(\rho,\varphi)} \frac{\partial(\rho,z)}{\partial(r',\alpha)} = -\rho \frac{\partial(\rho,z)}{\partial(r',\alpha)} \,. \tag{23}$$

Then, taking into account that

$$\begin{cases} \rho = R_0 + r' \cos\alpha \,, \\ z = r' \sin\alpha \,, \end{cases} \tag{24}$$

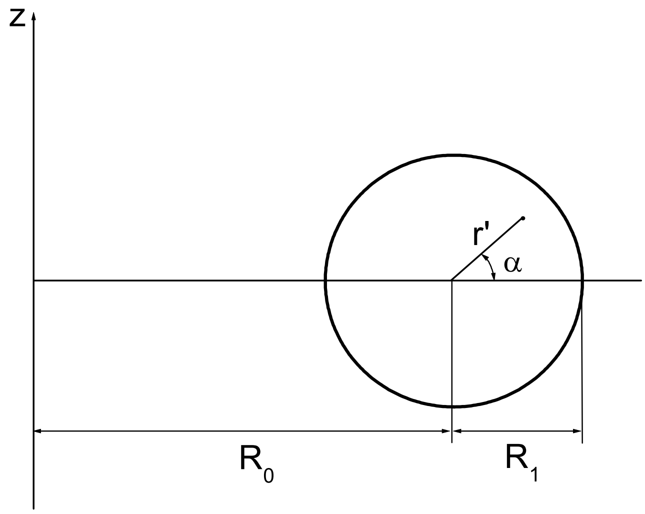

Figure 1: Cross-section of a torus by a half-plane $\varphi = \text{const}$.

we find

$$\frac{\partial(\rho, z)}{\partial(r', \alpha)} = r' , \tag{25}$$

and thus

$$\frac{\partial(x, y, z)}{\partial(r', \alpha, \varphi)} = -\rho r' = -r'(R_0 + r' \cos\alpha) . \tag{26}$$

Problem 35. Derive Eq. (26) by explicitly evaluating the Jacobian as the determinant of 3×3 matrix. Using the variables (r', α, φ) and the result (26), evaluate the volume of the torus.