Problem 22

Consider the evolution of a string of the length L, the ends of which are fixed. The positions of the ends are $x = -L/2$ and $x = L/2$. The motion of the string in the xy-plane is given by the wave equation

$$u_{tt} = c^2 u_{xx} , \tag{1}$$

for the amplitude $u = u(x,t)$, $x \in [-L/2, L/2]$, with the boundary conditions

$$u(-L/2,t) = u(L/2,t) = 0 . \tag{2}$$

The initial condition is

$$u(x,0) = 0 , \qquad u_t(x,0) = v_0[1-(2x/L)^2] . \tag{3}$$

[That is the initial amplitude is zero, but the initial velocity is finite, except for the end points.]

(a) Re-scale x, t, and u to get rid of letters in the equation, boundary conditions, and the initial condition. If you find it more convenient, you may also shift x.

(b) Make sure that the boundary conditions are canonical.

(c) Construct the orthonormal basis of the eigenfunctions of the Laplace operator in the space of functions obeying the boundary conditions (2).

(d) Find the solution $u(x,t)$ of the problem (1)-(3) in the form of the Fourier series in terms of the constructed ONB.

(e) Restore the original variables x, t, and u.

Problem 23

Consider the evolution of the temperature distribution in a rod of the length L, the ends of which are kept at zero temperature. The ends are located at the points $x = -L/2$ and $x = L/2$. The evolution of the temperature, $u = u(x,t)$, is given by the heat equation

$$u_t = \kappa u_{xx} , \tag{4}$$

(the thermal diffusivity κ is a positive constant) with the boundary conditions

$$u(-L/2, t) = u(L/2, t) = 0 \ . \tag{5}$$

The initial condition is

$$u(x, 0) = u_0[1 - (2x/L)^2] \ . \tag{6}$$

(a) Re-scale x, t, and u to get rid of letters in the equation, boundary conditions, and the initial condition. If you find it more convenient, you may also shift x.

(b) Make sure that the boundary conditions are canonical.

(c) Construct the orthonormal basis of the eigenfunctions of the Laplace operator in the space of functions obeying the boundary conditions (5).

(d) Find the solution $u(x, t)$ of the problem (4)-(6) in the form of the Fourier series in terms of the constructed ONB.

(e) Restore the original variables x, t, and u.

Some of the following integrals may prove helpful for solving the above two problems:

$$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x \, , \qquad \int \cos^2 x \, dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} x \, ,$$

$$\int x \sin x \, dx = \sin x - x \cos x \, , \qquad \int x \cos x \, dx = \cos x + x \sin x \, ,$$

$$\int x^2 \sin x \, dx = 2x \sin x - (x^2 - 2) \cos x \, ,$$

$$\int x^2 \cos x \, dx = 2x \cos x + (x^2 - 2) \sin x \, .$$