

Vector Calculus

Space Curves

A curve in three dimensional space can be specified as a vector function

$$\mathbf{r} \equiv \mathbf{r}(u) = (x(u), y(u), z(u)), \quad (1)$$

where \mathbf{r} is the radius vector and u is a real parameter of a quite arbitrary nature. If the curve of interest is a trajectory of a moving particle, then it is natural to use the time as the parameter u . If we are interested only in geometrical properties of the curve, then the most fundamental parameter—as will be shown below—is the arc length, l . In many geometrical cases it proves convenient to use as the parameter u just one of the Cartesian or polar coordinates.

Arc length. Given an arc of the curve $\mathbf{r}(u)$, $u \in [u_a, u_b]$, we want to know its length, l . By definition, the length of the curve is the limit of the lengths of the broken lines approximating our curve:

$$l = \lim_{N \rightarrow \infty} \sum_{j=1}^N |\Delta \mathbf{r}_j|, \quad (2)$$

where $\Delta \mathbf{r}_j = \mathbf{r}_j - \mathbf{r}_{j-1}$, and \mathbf{r}_j 's are (evenly separated) points on the curve, with $\mathbf{r}_0 = \mathbf{r}(u_a)$ and $\mathbf{r}_N = \mathbf{r}(u_b)$. The limit $N \rightarrow \infty$ implies $\Delta \mathbf{r}_j \rightarrow 0$, which allows us to write

$$\Delta \mathbf{r}_j \rightarrow \left. \frac{d\mathbf{r}}{du} \right|_{u=u_j} \Delta u_j, \quad (3)$$

because—by definition of the derivative of the radius vector with respect to u —we have

$$\frac{d\mathbf{r}}{du} = \lim_{\Delta u \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta u}. \quad (4)$$

We thus get

$$l = \lim_{N \rightarrow \infty} \sum_{j=1}^N \left| \left. \frac{d\mathbf{r}}{du} \right|_{u=u_j} \right| \Delta u_j. \quad (5)$$

Taking the limit corresponds to replacing summation with integration:

$$l = \int_{u_a}^{u_b} \left| \left. \frac{d\mathbf{r}}{du} \right| \right| du. \quad (6)$$

For practical usage of this formula it is important to remember that

$$\frac{d\mathbf{r}}{du} = \left(\frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du} \right), \quad (7)$$

and thus

$$\left| \frac{d\mathbf{r}}{du} \right| \equiv \sqrt{\frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du}} = \sqrt{\left(\frac{dx}{du} \right)^2 + \left(\frac{dy}{du} \right)^2 + \left(\frac{dz}{du} \right)^2}. \quad (8)$$

It is also convenient to introduce the differential of arc length

$$dl = \left| \frac{d\mathbf{r}}{du} \right| du = \sqrt{\frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du}} du, \quad (9)$$

so that

$$l = \int dl. \quad (10)$$

Natural parametrization. By natural parametrization one means using l as the parameter. If originally $u \neq l$, then, in order to get the natural parametrization, one has to express u as a function of l , by inverting the relation

$$l(u) = \int_{u_a}^u \left| \frac{d\mathbf{r}(u')}{du'} \right| du'. \quad (11)$$

Then $\mathbf{r}(l) \equiv \mathbf{r}(u(l))$.

Tangent vector. In the limit of $|\Delta\mathbf{r}| \rightarrow 0$, the vector $\Delta\mathbf{r} = \mathbf{r}(u + \Delta u) - \mathbf{r}(u)$ gets tangent to the curve at the point $\mathbf{r}(u)$. Hence, the vector

$$\frac{d\mathbf{r}}{du} = \lim_{\Delta u \rightarrow 0} \frac{\Delta\mathbf{r}}{\Delta u} \quad (12)$$

is tangent to the curve at the point $\mathbf{r}(u)$. In particular, the vector

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{dl} \quad (13)$$

is a tangent vector. An important property of the tangent $\hat{\mathbf{t}}$ is that its absolute value is always equal to unity. This property immediately follows from (9) upon replacing $u \rightarrow l$.

Curvature. The quantity

$$\kappa = \left| \frac{d\hat{\mathbf{t}}}{dl} \right| \quad (14)$$

is called *curvature*; its inverse $\rho = \kappa^{-1}$ is called *curvature radius*. [For an arc of a circle, the curvature is constant along the line, the curvature radius being just the radius of the circle.]

Principal normal and binormal. A remarkable property of the natural parametrization is that

$$\frac{d\hat{\mathbf{t}}}{dl} \perp \hat{\mathbf{t}}, \quad (15)$$

which is readily seen by differentiating with respect to l the above-established identity

$$\hat{\mathbf{t}} \cdot \hat{\mathbf{t}} = 1, \quad (16)$$

with the result

$$\frac{d\hat{\mathbf{t}}}{dl} \cdot \hat{\mathbf{t}} = 0. \quad (17)$$

The unit vector

$$\hat{\mathbf{n}} = \rho \frac{d\hat{\mathbf{t}}}{dl} \quad (18)$$

is called *principal normal*. The vector

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}, \quad (19)$$

which by construction is unit and perpendicular to both $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$, is called *binormal*.

Torsion. If the space curve is actually a plane curve—that is there is a plane containing the curve—then the vector $\hat{\mathbf{b}}$ is l -independent, being just the normal to the plane. Hence, the dependence of $\hat{\mathbf{b}}$ on l is crucial for distinguishing generic space curves from plane curves from, and the rate of this dependence is important for quantifying the difference. This rate is characterized by the quantity called *torsion*, τ , the absolute value of which is given by

$$|\tau| = \left| \frac{d\hat{\mathbf{b}}}{dl} \right|. \quad (20)$$

[Below we will see that it is also reasonable to introduce the sign of torsion by a certain rule.] Qualitatively, the idea behind Eq. (20) is close to the one behind Eq. (14). Correspondingly, the dimensionality of torsion is the same as the dimensionality of curvature, and it makes a perfect sense to introduce the *radius of torsion*, $\sigma = |\tau|^{-1}$.

An important property of the vector $d\hat{\mathbf{b}}/dl$ is that it always perpendicular to both $\hat{\mathbf{t}}$ and $\hat{\mathbf{b}}$, which means that it is parallel to $\hat{\mathbf{n}}$. To make sure that

$$\frac{d\hat{\mathbf{b}}}{dl} \perp \hat{\mathbf{b}} \quad (21)$$

we differentiate with respect to l the identity

$$\hat{\mathbf{b}} \cdot \hat{\mathbf{b}} = 1, \quad (22)$$

and get

$$\hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{b}}}{dl} = 0. \quad (23)$$

Similarly, differentiating the identity

$$\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} = 0, \quad (24)$$

we get

$$\frac{d\hat{\mathbf{b}}}{dl} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{dl} = 0. \quad (25)$$

Then we take into account that

$$\frac{d\hat{\mathbf{t}}}{dl} = \kappa \hat{\mathbf{n}}, \quad (26)$$

implying

$$\hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{dl} = 0, \quad (27)$$

and conclude that

$$\frac{d\hat{\mathbf{b}}}{dl} \cdot \hat{\mathbf{t}} = 0. \quad (28)$$

Having established that

$$\frac{d\hat{\mathbf{b}}}{dl} \parallel \hat{\mathbf{n}}, \quad (29)$$

and recalling the definition (20) of the absolute value of torsion, we now can define the torsion by the relation

$$\frac{d\hat{\mathbf{b}}}{dl} = -\tau \hat{\mathbf{n}}, \quad (30)$$

in which the negative sign is a matter of convention. The convenience of having an algebraic torsion instead of positive definite one is clear from the following example. A helix is a curve which features a

constant curvature and a constant torsion. If we fix the value of curvature (which is positive definite by definition) and the absolute value of torsion, there are still two different types of helices: right- and left-handed ones. With torsion being an algebraic quantity, the two differ by the sign of torsion. This is the general rule: The sign of the torsion is associated with the (local) handedness (or chirality) of the curve.

We have revealed an important differential-geometric meaning of the two derivatives: $d\hat{\mathbf{t}}/dl$ and $d\hat{\mathbf{b}}/dl$.—What about the derivative $d\hat{\mathbf{n}}/dl$? Differentiating the relation

$$\hat{\mathbf{n}} = \hat{\mathbf{b}} \times \hat{\mathbf{t}}, \quad (31)$$

we get

$$\frac{d\hat{\mathbf{n}}}{dl} = \frac{d\hat{\mathbf{b}}}{dl} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \frac{d\hat{\mathbf{t}}}{dl} = -\tau\hat{\mathbf{n}} \times \hat{\mathbf{t}} + \kappa\hat{\mathbf{b}} \times \hat{\mathbf{n}} = \tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{t}}. \quad (32)$$

Combining all the three derivatives, we arrive at a system of differential equations

$$\begin{cases} d\hat{\mathbf{t}}/dl = \kappa\hat{\mathbf{n}}, \\ d\hat{\mathbf{n}}/dl = \tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{t}}, \\ d\hat{\mathbf{b}}/dl = -\tau\hat{\mathbf{n}}, \end{cases} \quad (33)$$

known as *Frenet-Serret* formulae. We see that the curvature and torsion completely define the geometric shape of the curve, since all the three vectors, $\hat{\mathbf{t}}(l)$, $\hat{\mathbf{n}}(l)$, $\hat{\mathbf{b}}(l)$ can be restored from the pair of functions $(\kappa(l), \tau(l))$ by solving the system of Frenet-Serret equations, and then obtaining the radius vector $\mathbf{r}(l)$ by a simple integration:

$$\mathbf{r}(l) = \mathbf{r}_0 + \int_0^l \hat{\mathbf{t}}(l') dl'. \quad (34)$$

The position of the beginning of the curve, $\mathbf{r}_0 \equiv \mathbf{r}(0)$, as well as the spatial orientation of the curve, fixed by $\hat{\mathbf{t}}(0)$, $\hat{\mathbf{n}}(0)$, $\hat{\mathbf{b}}(0)$, come as free constants of integration. These quantities, however, are irrelevant to the shape of the curve.

Surfaces

A surface in three dimensional space can be specified as a vector function of two independent real variables:

$$\mathbf{r} \equiv \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)). \quad (35)$$

This formula maps some area in the uv -plane onto a surface in a three-dimensional space. The nature of the variables u and v can be quite arbitrary. For example, the two can be just x - and y -coordinates, in which case we have

$$\mathbf{r} \equiv \mathbf{r}(x, y) = (x, y, z(x, y)). \quad (36)$$

For a closed surface like a sphere—say, the surface of the Earth—it is quite natural to use spherical angles θ and φ , in which case the surface is parameterized as

$$\mathbf{r} \equiv \mathbf{r}(\theta, \varphi) = (x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi)). \quad (37)$$

In this course, we confine ourselves with considering the issues relevant to the surface integrals, and, in particular, to evaluation of the surface area.

Let us fix on a surface some point $\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$. We can readily construct two space curves, each of them lying in the surface and containing the point \mathbf{r}_0 . The first curve is

$$\mathbf{r}(u) \equiv \mathbf{r}(u, v_0) , \quad (38)$$

with u understood as a parameter of the curve, and the second curve is

$$\mathbf{r}(v) \equiv \mathbf{r}(u_0, v) , \quad (39)$$

where the parameter of the curve is v . The above-developed theory of space curves then yields two tangent vectors:

$$\mathbf{t}_1 = \frac{\partial \mathbf{r}}{\partial u} \quad (40)$$

and

$$\mathbf{t}_2 = \frac{\partial \mathbf{r}}{\partial v} , \quad (41)$$

which are the tangents of the two curves at the point \mathbf{r}_0 . The two tangent vectors define a *tangent plane*. By definition, a *normal* to the surface at the point \mathbf{r}_0 is any vector normal to the tangent plane at this point. Since the tangent plane is defined by the tangent vectors \mathbf{t}_1 and \mathbf{t}_2 , the vector

$$\mathbf{n} = \mathbf{t}_1 \times \mathbf{t}_2 = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \quad (42)$$

is a normal.

Surface area. The two infinitesimal vectors,

$$d\mathbf{l}_1 = \mathbf{t}_1 du \quad \text{and} \quad d\mathbf{l}_2 = \mathbf{t}_2 dv , \quad (43)$$

define an infinitesimal parallelogram on the surface, with the area

$$dS = |d\mathbf{l}_1 \times d\mathbf{l}_2| = |\mathbf{t}_1 \times \mathbf{t}_2| du dv = |\mathbf{n}| du dv = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv . \quad (44)$$

Hence, splitting the whole surface into infinitesimal parallelograms, we naturally define the surface area as the integral

$$S = \int dS = \int \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv . \quad (45)$$

Line and Surface Integrals

In the previous sub-sections we have introduced infinitesimal elements of an arc and surface with the arch length and surfaces area differentials given by Eqs. (9) and (44), respectively. We can also introduce *vector* infinitesimal elements of arc and surface by

$$\vec{dl} = \frac{d\mathbf{r}}{du} du , \quad (46)$$

$$\vec{dS} = \mathbf{n} du dv = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} du dv . \quad (47)$$

With this vector elements, one can construct *line* and *surface* integrals from vector and scalar fields. Below we will consider only the case of vector fields, since the case of a scalar is quite straightforward. Given a vector field $\mathbf{A}(\mathbf{r})$ and a curve C , we can construct the following three line integrals:

$$\int_C \mathbf{A}(\mathbf{r}) dl, \quad \int_C \mathbf{A}(\mathbf{r}) \cdot d\vec{l}, \quad \int_C \mathbf{A}(\mathbf{r}) \times d\vec{l}. \quad (48)$$

Clearly, the first and the third integrals are vectors, while the second integral is a scalar.

Analogously, with a vector field $\mathbf{A}(\mathbf{r})$ and a surface S , we construct the following three surface integrals:

$$\int_S \mathbf{A}(\mathbf{r}) dS, \quad \int_S \mathbf{A}(\mathbf{r}) \cdot d\vec{S}, \quad \int_S \mathbf{A}(\mathbf{r}) \times d\vec{S}. \quad (49)$$

Once again, the first and the third integrals are vectors, while the second integral is a scalar.

Note that while an explicit formula for evaluating each of these integrals depends on the parametrization of the curve/surface, the result is parametrization independent, since definitions of the integrals are invariant with respect to the choice of the parametrization.

As an important example of line integral, consider the Biot-Savart law for the magnetic field \mathbf{B} created by a constant current I in a wire of the shape C :

$$\mathbf{B}(\mathbf{r}_0) = I \int_C \frac{d\vec{l} \times (\mathbf{r}_0 - \mathbf{r})}{|\mathbf{r}_0 - \mathbf{r}|^3}. \quad (50)$$

Here \mathbf{r}_0 is the (fixed) radius-vector of the point where the magnetic field is being calculated, while the vector \mathbf{r} runs along the curve C .

Gradient, Divergence, Curl

Let us introduce a vector operator (so-called nabla-operator)

$$\nabla = (\partial_x, \partial_y, \partial_z), \quad (51)$$

where $\partial_x \equiv \partial/\partial x$, $\partial_y \equiv \partial/\partial y$, $\partial_z \equiv \partial/\partial z$. By definition, the operator ∇ acts on scalar and vector fields, and the way it acts is reminiscent of the operations with vectors and scalars, in which the operator ∇ behaves like a vector. Acting on a scalar field $\Phi(\mathbf{r})$, nabla-operator produces a vector field

$$\nabla\Phi = (\partial_x\Phi, \partial_y\Phi, \partial_z\Phi). \quad (52)$$

This field is called *gradient* of the field $\Phi(\mathbf{r})$, and is denoted as $\text{grad } \Phi$. That is $\nabla\Phi \equiv \text{grad } \Phi$.

There are two ways for nabla-operator to act on a vector field $\mathbf{A}(\mathbf{r})$. The first way is to form a scalar, an analog of the inner product:

$$\nabla \cdot \mathbf{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z. \quad (53)$$

This scalar is called *divergence* of the vector field $\mathbf{A}(\mathbf{r})$ and is denoted as $\text{div } \mathbf{A}$. That is $\nabla \cdot \mathbf{A} \equiv \text{div } \mathbf{A}$. The second way is to form a vector, an analog of the vector cross-product:

$$\nabla \times \mathbf{A} = \hat{e}_x(\partial_y A_z - \partial_z A_y) + \hat{e}_y(\partial_z A_x - \partial_x A_z) + \hat{e}_z(\partial_x A_y - \partial_y A_x). \quad (54)$$

This vector is called *curl* of the vector field $\mathbf{A}(\mathbf{r})$ and is denoted as $\text{curl } \mathbf{A}$. That is $\nabla \times \mathbf{A} \equiv \text{curl } \mathbf{A}$. (Along with the term *curl* they also use the term *rotor*, and, correspondingly, use the notation $\text{rot } \mathbf{A}$ for $\nabla \times \mathbf{A}$.)

All the three differential operations are extremely important for physics applications. Correspondingly, also important are the following three theorems that involve these operations. (The proofs of the theorems go beyond the scope of this course.)

Integral from a gradient. The following relation is true for any single-valued scalar field Φ and any curve C_{ab} that starts at the point \mathbf{r}_a and ends at the point \mathbf{r}_b :

$$\int_{C_{ab}} \nabla\Phi \cdot d\vec{l} = \Phi(\mathbf{r}_b) - \Phi(\mathbf{r}_a) . \quad (55)$$

Note that the value of the integral turns out to be independent of the shape of the curve connecting the two points. In particular, for a closed contour C we get

$$\oint_C \nabla\Phi \cdot d\vec{l} = 0 . \quad (56)$$

As a typical example, consider the work of a *potential force*. For any force \mathbf{F} , the work along the path C_{ab} is given by the line integral

$$\text{Work} = \int_{C_{ab}} \mathbf{F} \cdot d\vec{l} . \quad (57)$$

Now a *potential force* is the force which can be expressed as the gradient of a certain scalar field U , called potential:

$$\mathbf{F} = -\nabla U . \quad (58)$$

[The sign in this formula is just the matter of definition.] For a potential force we thus have:

$$\text{Work} = U(\mathbf{r}_a) - U(\mathbf{r}_b) . \quad (59)$$

That is the work of a potential force is independent of the shape of the path, being equal to the difference of the potentials at two ends of the path. In particular, the work of a potential force along a closed path is zero.

Gauss-Ostrogradsky theorem states that for any closed surface S and an arbitrary vector field $\mathbf{A}(\mathbf{r})$ the following is true

$$\oint_S \mathbf{A} \cdot d\vec{S} = \int_V \text{div}\mathbf{A} dV , \quad (60)$$

where the bulk integral is over the region enclosed by the surface S . [The direction of the vector $d\vec{S}$ is supposed to be outside the surface, otherwise there appears the sign minus.] The surface integral of this type is called a flux of the vector field \mathbf{A} across the surface S .

As an example, consider electrostatics. From the Gauss law of electrostatics we have

$$\text{div}\mathbf{E} = \rho , \quad (61)$$

where \mathbf{E} is the electric field and $\rho(\mathbf{r})$ is the charge density. (Normally, there is also a proportionality coefficient, which is not important for our purposes.) From the Gauss-Ostrogradsky theorem we then get

$$\oint_S \mathbf{E} \cdot d\vec{S} = \int_V \rho dV . \quad (62)$$

By the definition of ρ , the r.h.s. yields the charge enclosed by the surface S . Hence, we arrive at a general conclusion that a flux of electric field across a closed surface is always equal to the charge enclosed by the surface.

Stokes theorem states that for any closed contour C and an arbitrary vector field $\mathbf{A}(\mathbf{r})$ the following is true

$$\oint_C \mathbf{A} \cdot d\vec{l} = \int_S \text{curl } \mathbf{A} \cdot d\vec{S}, \quad (63)$$

where S is *any* surface for which the contour C plays the role of boundary. [The sign of the vector $d\vec{S}$ is fixed by the right-hand rule with respect to the direction of integration along the contour C .] Any linear integral of this type is called circulation of the vector \mathbf{A} along the contour C .

As an example, consider magnetostatics where the magnetic field $\mathbf{B}(\mathbf{r})$ created by time-independent currents always obeys the law (up to a constant coefficient which is not important here)

$$\text{curl } \mathbf{B} = \mathbf{j}, \quad (64)$$

where $\mathbf{j}(\mathbf{r})$ is the so-called current density. The physical meaning of the current density is as follows. For any surface S , the integral

$$\int_S \mathbf{j} \cdot d\vec{S} \quad (65)$$

yields the electric current through the surface.

From the Stokes theorem we then have

$$\oint_C \mathbf{B} \cdot d\vec{l} = \int_S \mathbf{j} \cdot d\vec{S}. \quad (66)$$

Hence, the circulation of magnetic field along a contour C is always equal to the electric current through this contour.