## **One-Dimensional Wave Equation on an Infinite Axis**

Let us analyze one-dimensional wave equation on an infinite axis

$$u_{tt} = c^2 u_{xx}, \qquad x \in (-\infty, \infty).$$
(1)

We assume that at t = 0 we know the initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x).$$
 (2)

In principle, we could use the Fourier integral method. It turns out, however, that for Eq. (1) there exists a very special and simple treatment, based on the fact that for any function f(x), each of the following two functions,

$$u^{(+)}(x,t) = f(x-ct), \qquad u^{(-)}(x,t) = f(x+ct), \qquad (3)$$

yield a solution. (Which is directly checked by the chain rule!) These solutions are the running waves. The wave  $u^{(+)}$  runs in the positive direction, while the wave  $u^{(-)}$  runs in the negative direction. In view of the linearity of the wave equation, a sum of any two solutions is also a solution, and we arrive at an idea that the general solution to Eq. (1) can be represented as

$$u(x,t) = f(x - ct) + g(x + ct) , \qquad (4)$$

where f and g any two functions. Indeed, we have two given functions,  $u_0(x)$  and  $v_0(x)$ , and we expect that the freedom of choosing two functions, f(x) and g(x), is just what we need to satisfy the two initial conditions (2). Clearly, to satisfy the initial conditions it is necessary and sufficient to satisfy the two equations:

$$f(x) + g(x) = u_0(x)$$
, (5)

$$-cf'(x) + cg'(x) = v_0(x) . (6)$$

Integrating the second equation, we get

$$-cf(x) + cg(x) = \int_{-\infty}^{x} v_0(x') \, dx' + A \,, \tag{7}$$

where A is some constant. Multiplying (5) by c and then forming the sum and the difference with (7), we get

$$f(x) = \frac{u_0(x)}{2} - \frac{1}{2c} \int_{-\infty}^x v_0(x') \, dx' - A_0 \,, \tag{8}$$

$$g(x) = \frac{u_0(x)}{2} + \frac{1}{2c} \int_{-\infty}^x v_0(x') \, dx' + A_0 \,, \tag{9}$$

where the constant  $A_0 = A/2c$  remains undefined, but this is because it drops out from the final answer:

$$u(x,t) = \frac{u_0(x-ct)}{2} + \frac{u_0(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(x') \, dx' \,. \tag{10}$$

Note that the solution is especially simple if  $v_0 \equiv 0$ . The initial profile just splits into two identical running waves the shape of which is identical to that of the initial state, but the amplitude is smaller by a factor of 2. Also worth noting is the *causality* property: At a given time moment t, the initial condition at the point  $x_0$  can effect the solution u(x,t) only within the causality interval  $x \in [x_0 - ct, x_0 + ct]$ . That is a signal cannot propagate faster than the speed c.