Hilbert Spaces

Hilbert space is a vector space with some extra structure. We start with formal (axiomatic) definition of a vector space.

Vector Space. Vector space, ν, over the field of complex numbers, $\mathcal{C}$, is a set of elements $|a\rangle$, $|b\rangle, \ldots$, satisfying the following axioms.

$\diamond$ For each two vectors $|a\rangle$, $|b\rangle \in \nu$ there exists a summation procedure: $|a\rangle + |b\rangle = |c\rangle$, where $|c\rangle \in \nu$. The summation obeys the following laws.

$$|a\rangle + |b\rangle = |b\rangle + |a\rangle \quad \text{(commutative)} \; , \tag{1}$$

$$|a\rangle + (|b\rangle + |c\rangle) = (|a\rangle + |b\rangle) + |c\rangle \quad \text{(associative)} \; . \tag{2}$$

$\diamond$ There exists a zero vector $|0\rangle$, such that $\forall \, |a\rangle$:

$$|a\rangle + |0\rangle = |a\rangle \; . \tag{3}$$

$\diamond$ $\forall \, |a\rangle \quad \exists \, |{-a}\rangle$ (additive inverse) such that

$$|a\rangle + |{-a}\rangle = |0\rangle \; . \tag{4}$$

[Here we start using the symbol $\forall$ that means 'for all' ('for each', 'for any') and the symbol $\exists$ that means 'there exists'.]

$\diamond$ There exists a procedure of multiplication by a scalar $\alpha \in \mathcal{C}$. That is $\forall \, |a\rangle \in \nu$, $\forall \, \alpha \in \mathcal{C}$: $\exists \; \alpha \, |a\rangle \in \nu$. Multiplication by a scalar obeys the following laws.

$$\alpha \, (\beta \, |a\rangle \,) = (\alpha\beta) \, |a\rangle \; , \tag{5}$$

$$1 \cdot |a\rangle = |a\rangle \; , \tag{6}$$

$$\alpha(|a\rangle + |b\rangle) = \alpha|a\rangle + \alpha|b\rangle \; , \tag{7}$$

$$(\alpha + \beta)|a\rangle = \alpha|a\rangle + \beta|a\rangle \; . \tag{8}$$

From the above axioms it follows that $\forall \, |a\rangle$

$$0 \cdot |a\rangle = |0\rangle \; , \tag{9}$$

$$(-1) \cdot |\, a\rangle = |-a\rangle \ . \tag{10}$$

Problem 21. On the basis of the axioms:
(a) Show that the zero element is unique.
(b) Show that for any vector $|a\rangle$ there exists only one additive inverse.
(c) Show that for any vector $|a\rangle$ the relation $|a\rangle + |x\rangle = |a\rangle$ implies that $|x\rangle = |0\rangle$.
(d) Derive (9)-(10).
Important (!): Here you are not allowed to use the subtraction procedure, which is not defined yet, and *cannot* be defined *prior* to establishing the uniqueness of additive inverse.

Once the uniqueness of additive inverse is established (Problem 21), it is convenient to define subtraction as simply adding additive inverse:

$$|\, a\rangle - |\, b\rangle \ \equiv \ |\, a\rangle + |-b\rangle \ . \tag{11}$$

Inner product. Now let us formally introduce the inner-product vector space as a vector space in which for any two vectors $|a\rangle$ and $|b\rangle$ there exists the inner product, $\langle\, b\,|\, a\,\rangle$, which is a complex-valued function of the two vectors satisfying the following properties.

$$\langle\, b\,|\, a\,\rangle = \overline{\langle\, a\,|\, b\,\rangle} \ . \tag{12}$$

Here the bar denotes complex conjugation. From Eq. (12) it directly follows that the inner product of a vector with itself is always real. The next axiom of inner product requires also that $\langle a|a\rangle$ be *positive* if $|a\rangle \neq |0\rangle$, and that $\langle\, 0\,|\, 0\,\rangle = 0$. Finally, we require that

$$\langle\, a\,|\, \alpha\, u + \beta\, v\,\rangle \ = \ \alpha\,\langle\, a\,|\, u\rangle \ + \ \beta\,\langle\, a\,|\, v\rangle \ . \tag{13}$$

Here we use a convenient notation $|\, u +\ v\,\rangle \equiv |u\rangle + |v\rangle$ and $|\, \alpha\, u\,\rangle \equiv \alpha\,|u\rangle$. From Eqs. (12) and (13) it *follows* that

$$\langle\, \alpha\, u + \beta\, v\,|\, a\,\rangle \ = \ \alpha^*\,\langle\, u\,|\, a\rangle \ + \ \beta^*\,\langle\, v\,|\, a\rangle \ . \tag{14}$$

Linear self-adjoint operator. Linear operator, L, is a function from ν to ν—that is it transforms one vector of the given space into another vector of the same space—that satisfies the following requirement:

$$L\,|\,\alpha\,u+\beta\,v\,\rangle \;=\; \alpha L\,|u\rangle+\beta L\,|v\rangle\;. \tag{15}$$

The operator $L^\dagger$ is called adjoint to L if $\forall\,|a\rangle,|b\rangle\in\nu$:

$$\langle\,b\,|\,L\,|\,a\,\rangle=\overline{\langle\,a\,|\,L^\dagger\,|\,b\,\rangle}\;. \tag{16}$$

With a convenient notation $|\,La\,\rangle\equiv\; L\,|\,a\,\rangle$ and taking into account the axiom (12) we can rewrite (16) as

$$\langle\,b\,|\,La\,\rangle=\langle\,L^\dagger b\,|\,a\,\rangle\;. \tag{17}$$

Clearly, $(L^\dagger)^\dagger\equiv L$.

An operator L is self-adjoint (Hermitian) if $L^\dagger=L$, that is if $\forall\,|a\rangle,|b\rangle$:

$$\langle\,b\,|\,La\,\rangle=\langle\,Lb\,|\,a\,\rangle\;\;. \tag{18}$$

A vector $|u\rangle$ is called eigenvector of the operator L, if

$$L|u\rangle=\lambda_u|u\rangle\;, \tag{19}$$

where λ_u is some number which is called eigenvalue. For a self-adjoint operator all the eigenvalues are real. To make it sure we first observe that from (18) it follows that for *any* vector $|a\rangle$ the quantity $\langle\,a\,|\,L\,|\,a\,\rangle$ is real provided L is self-adjoint. And we just need to note that $\lambda_u=\langle\,u\,|\,L\,|\,u\,\rangle/\langle\,u\,|\,u\,\rangle$.

Two vectors of an inner-product vector space are called orthogonal if their inner product is zero.

Theorem. Any two eigenvectors, $|u_1\rangle$ and $|u_2\rangle$, of the linear self-adjoint operator L are orthogonal if their eigenvalues are different.

Proof. From $L|u_1\rangle=\lambda_{u_1}|u_1\rangle$ and $L|u_2\rangle=\lambda_{u_1}|u_2\rangle$ we get

$$\langle u_2|L|u_1\rangle=\lambda_{u_1}\langle u_2|u_1\rangle\;, \tag{20}$$

$$\langle u_1|L|u_2\rangle=\lambda_{u_2}\langle u_1|u_2\rangle\;. \tag{21}$$

Complex-conjugating the latter equality and taking into account (12), (18), and the the fact that eigenvalues of a Hermitian operator are real, we obtain

$$\langle u_2|L|u_1\rangle=\lambda_{u_2}\langle u_2|u_1\rangle\;. \tag{22}$$

Comparing this to (20) we have

$$\lambda_{u_1} \langle u_2|u_1\rangle = \lambda_{u_2} \langle u_2|u_1\rangle \ , \tag{23}$$

which given $\lambda_{u_1} \neq \lambda_{u_2}$ implies

$$\langle u_2|u_1\rangle = 0 \ . \tag{24}$$

What if $\lambda_{u_1} = \lambda_{u_2}$? In this case $|u_1\rangle$ and $|u_2\rangle$ are not necessarily orthogonal. What is important, however, is that all the eigenvectors of the same eigenvalue λ form a *vector sub-space* of the original vector space, because any linear combination of two eigenvectors with one and the same eigenvalue λ makes another eigenvector with the same eigenvalue λ. Within this subspace one can choose an orthogonal basis. This way we arrive at an orthogonal basis of the eigenvectors of the self-adjoint operator, which is very useful for many practical applications.

Finite-dimensional inner-product vector space. Suppose we have a set of n vectors $\{|\phi_j\rangle\}$, $(j = 1, 2, \ldots, n)$ of a vector space ν. The following vector,

$$|a\rangle \ = \sum_{j=1}^{n} c_j \, |\phi_j\rangle \ \ , \tag{25}$$

where $\{c_j\}$ are some complex numbers, is called a *linear combination* of the vectors $\{|\phi_j\rangle\}$. As is directly seen from the axioms of the vector space, all the linear combinations of the given set $\{|\phi_j\rangle\}$ form some vector space $\tilde{\nu}$—for such a space we will be using the term *sub-space* to stress the fact that $\tilde{\nu} \subseteq \nu$. If $\tilde{\nu} = \nu$, that is if *any* vector of the space ν can be represented as a linear combination of the vectors $\{|\phi_j\rangle\}$, then the set $\{|\phi_j\rangle\}$ is called *basis* and the space ν is a *finite-dimensional* space. For any finite-dimensional vector space there is a minimal possible number n for a vector set to form a basis. This number is called *dimensionality* of the space, and, correspondingly, the space is called n-dimensional space. In an n-dimensional vector space, all the vectors of an n-vector basis $\{|\phi_j\rangle\}$ are *linear independent* in the sense that any linear combination of these vectors is different from zero if at least one of the numbers c_j is non-zero. Indeed, if the vectors $\{|\phi_j\rangle\}$ were linear dependent, that is if $c_j \neq 0$ for at least one $j = j_0$, then the vector $|\phi_{j_0}\rangle$ could be expressed as a linear combination of the other vectors of the basis, and we would get a basis of $(n-1)$ vectors—all the vectors of the original basis, but the vector $|\phi_{j_0}\rangle$. But this is impossible by definition of the dimensionality.

Hence, without loss of generality we may deal only with a linear-independent basis, since we can always eliminate linear dependent vectors. Below we assume that our basis is linear independent, so that n is the dimensionality of our space.

Given an arbitrary basis$\{|\phi_j\rangle\}$, we can construct an *orthonormal* basis $\{|e_j\rangle\}$:

$$\langle\, e_i \,|\, e_j \,\rangle = \delta_{ij} \; . \tag{26}$$

[The term *orthonormal* means *orthogonal and normal*, where *orthogonal* means $\langle\, e_i \,|\, e_j \,\rangle = 0$, $\forall i \neq j$, and *normal* means $\langle\, e_i \,|\, e_i \,\rangle = 1$, $\forall i$.] This is done by the Gram-Schmidt orthonormalization procedure:

$$\begin{aligned}
|\tilde{e}_1\rangle &= |\phi_1\rangle \; , & |e_1\rangle &= |\tilde{e}_1\rangle / \sqrt{\langle \tilde{e}_1 | \tilde{e}_1 \rangle} \; , \\
|\tilde{e}_2\rangle &= |\phi_2\rangle - \langle e_1 | \phi_2 \rangle \, |e_1\rangle \; , & |e_2\rangle &= |\tilde{e}_2\rangle / \sqrt{\langle \tilde{e}_2 | \tilde{e}_2 \rangle} \; , \\
|\tilde{e}_3\rangle &= |\phi_3\rangle - \langle e_1 | \phi_3 \rangle \, |e_1\rangle - \langle e_2 | \phi_3 \rangle \, |e_2\rangle \; , & |e_3\rangle &= |\tilde{e}_3\rangle / \sqrt{\langle \tilde{e}_3 | \tilde{e}_3 \rangle} \; , \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \\
|\tilde{e}_n\rangle &= |\phi_n\rangle - \langle e_1 | \phi_n \rangle \, |e_1\rangle - \ldots - \langle e_{n-1} | \phi_n \rangle \, |e_{n-1}\rangle \; , & |e_n\rangle &= |\tilde{e}_n\rangle / \sqrt{\langle \tilde{e}_n | \tilde{e}_n \rangle} \; .
\end{aligned}$$

By construction, each successive vector $|\tilde{e}_{j_0}\rangle$ is orthogonal to all previous vectors $|e_j\rangle$, and then it is just properly normalized to yield $|e_{j_0}\rangle$.

With the tool of the orthonormal basis, it is easy to show that each n-dimensional inner-product space is *isomorphic*—the precise meaning of this word will become clear later—to the vector space $\mathcal{C}^n$ of the complex-number rows, $(x_1, x_2, \ldots, x_n)$, where the summation, multiplication by a complex number, and inner product are defined as follows.

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, \, x_2 + y_2, \, \ldots, \, x_n + y_n) \, , \tag{27}$$

$$\alpha \, (x_1, x_2, \ldots, x_n) = (\alpha \, x_1, \, \alpha \, x_2, \, \ldots, \, \alpha \, x_n) \, , \tag{28}$$

$$(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = x_1^* y_1 + x_2^* y_2 + \ldots + x_n^* y_n \, . \tag{29}$$

Fixing some orthonormal basis $\{|e_j\rangle\}$ in our vector space ν, we first note that $\forall \, |x\rangle \in \nu$, the coefficients x_j—referred to as components of the vector $|x\rangle$ with respect to the given basis—in the expansion

$$|x\rangle = \sum_{j=1}^{n} x_j \, |e_j\rangle \; , \tag{30}$$

are unique, being given by

$$x_j = \langle e_j | x \rangle \; , \tag{31}$$

which is seen by constructing inner products of the r.h.s. of (30) with the basis vectors. This leads to a one-to-one mapping between the vectors of the spaces ν and $\mathcal{C}^n$:

$$|x\rangle \leftrightarrow (x_1, x_2, \ldots, x_n)\,. \tag{32}$$

By the axioms of the inner product, from (30)-(31) one makes sure that if $|x\rangle \leftrightarrow (x_1, x_2, \ldots, x_n)$ and $|y\rangle \leftrightarrow (y_1, y_2, \ldots, y_n)$, then

$$|x\rangle + |y\rangle \leftrightarrow (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)\,, \tag{33}$$

$$\alpha\,|x\rangle \leftrightarrow \alpha\,(x_1, x_2, \ldots, x_n)\,, \tag{34}$$

$$\langle x|y\rangle = (x_1, x_2, \ldots, x_n)\cdot(y_1, y_2, \ldots, y_n)\,, \tag{35}$$

That is we see a complete equivalence of the two vector spaces, and it is precisely this type of equivalence which we understand by the word isomorphism.

Normed vector space. A vector space ν is said to be normed if $\forall\, x \in \nu$ there is defined a non-negative real number $\|x\|$ satisfying the following requirements (axioms of norm).

$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(Triangle inequality)}\ . \tag{36}$$

$\forall \alpha \in \mathcal{C},\ \forall\, x \in \nu$:

$$\|\alpha x\| = |\alpha| \cdot \|x\| \qquad \text{(Linearity of the norm)}\ . \tag{37}$$

Note that the linearity of the norm implies $\|0\| = 0$.

Finally, we require that from $\|x\| = 0$ it should follow that $x = 0$. Hence, the norm is positive for all vectors, but zero.

Problem 22. Consider the vector space R^2, i.e. the set of pairs (x, y) of real numbers.
(a) Show that the function $\|\cdot\|_M$ defined by $\|(x, y)\|_M = \max\{|x|, |y|\}$ is a norm on R^2.
(b) Show that the function $\|\cdot\|_S$ defined by $\|(x, y)\|_S = |x| + |y|$ is a norm on R^2.
(c) In any normed space ν the unit ball $\mathcal{B}_1$ is defined to be $\{u \in \nu \mid \|u\| \leq 1\}$. Draw the unit ball in R^2 for each of the norms, $\|\cdot\|_M$ and $\|\cdot\|_S$.

An inner-product vector space is *automatically* a normed vector space, if one defines the norm as

$$\|x\| = \sqrt{\langle\, x \,|\, x\,\rangle}\ . \tag{38}$$

Problem 23. Prove that with the definition (38): (i) all the axioms of norm are satisfied and (ii) there take place the following specific to the inner-product norm properties.

$$\langle\, y \,|\, x \,\rangle \;=\; 0 \qquad \Rightarrow \qquad \|x+y\|^2 \;=\; \|x\|^2 + \|y\|^2 \;, \tag{39}$$

Cauchy-Bunyakovsky-Schwarz inequality:

$$\mathrm{Re}\,\langle\, y \,|\, x \,\rangle \;\leq\; \|x\| \cdot \|y\| \;, \tag{40}$$

moreover,

$$|\langle\, y \,|\, x \,\rangle| \;\leq\; \|x\| \cdot \|y\| \;, \tag{41}$$

parallelogram law:

$$\|x-y\|^2 + \|x+y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2 \;, \tag{42}$$

and "polar identity"

$$\langle\, y \,|\, x \,\rangle = (1/4)\left\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right\} \;. \tag{43}$$

Hints. The triangle inequality directly follows from the simplest version of Cauchy-Bunyakovsky inequality, Eq. (40). To prove Eq. (40), utilize the fact that the product $\langle\, x+\lambda y \,|\, x+\lambda y \,\rangle$ is a second order polynomial of λ which is non-negative $\forall \lambda$ (by an axiom of the inner product), which implies a certain constraint on its discriminant. To arrive at Eq. (41), use the same approach, but with $\lambda \to \lambda\langle\, y \,|\, x \,\rangle$.

Convergent sequence. Let ν be a normed vector space. The sequence of vectors $\{x_k,\ k=1,2,3,\ldots\} \in \nu$ is said to be convergent to the vector $x \in \nu$, if $\forall\, \varepsilon > 0\ \exists\, k_\varepsilon$, such that if $k > k_\varepsilon$, then $\|x - x_k\| < \varepsilon$. The fact that the sequence $\{x_k\}$ converges to x is symbolically written as $x = \lim_{k\to\infty} x_k$.

Cauchy sequence. The sequence of vectors $\{x_k,\ k=1,2,3,\ldots\} \in \nu$ is said to be a Cauchy sequence if $\forall\, \varepsilon > 0\ \exists\, k_\varepsilon$, such that if $m,n > k_\varepsilon$, then $\|x_m - x_n\| < \varepsilon$.

Any convergent sequence is necessarily a Cauchy sequence.

Problem 24. Show this.

The question now is: Does *any* Cauchy sequence in a given vector space ν converge to some vector $x \in \nu$? The answer is not necessarily positive and essentially depends on the structure of the vector space ν. (Normed spaces where all Cauchy sequences converge are called *complete spaces*, or Banach spaces.) For any inner-product vector space of a finite dimension the answer is positive and is readily proven by utilizing the fact of existence of the orthonormal basis, $\{|e_i\rangle\}$. First we note that if

$$| a \rangle = \sum_{i=1}^{n} \alpha_i | e_i \rangle \, , \tag{44}$$

then

$$\|a\| = \sqrt{|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_n|^2} \, . \tag{45}$$

If we have a Cauchy sequence of vectors $|a^{(k)}\rangle$, then for any given $i = 1, 2, \cdots, n$ the i-th coordinates of the vectors, $\alpha_i^{(k)}$, form a Cauchy sequence of complex numbers. Any Cauchy sequence of complex numbers converges to some complex number.—This is a consequence of the fact that a complex-number Cauchy sequence is equivalent to two real-number Cauchy sequences (for the real and imaginary parts, respectively) and a well-known fact of the theory of real numbers that any real-number Cauchy sequence is convergent (completeness of the set of real numbers). We thus introduce the numbers

$$\alpha_i = \lim_{k \to \infty} \alpha_i^{(k)} \, , \tag{46}$$

and easily see that our vector sequence converges to the vector

$$| a \rangle = \sum_{i=1}^{n} \alpha_i | e_i \rangle \, . \tag{47}$$

A *complete inner-product space* is called *Hilbert space*. We have demonstrated that *all* finite-dimensional inner-product vector spaces are Hilbert spaces. What about infinite-dimensional ones?

An important example of complete infinite-dimensional vector space is the space $C[a, b]$ of all continuous complex-valued functions $f(x)$, $x \in [a, b]$ with the norm defined as $\|f\|_{\text{sup}} = \max\{|f(x)|, \; x \in [a, b]\}$. This norm (which is called 'sup' norm, from 'supremum') guaranties that any Cauchy sequence of functions $f_k(x)$ converges *homogeneously* to some continuous function $f(x)$. A problem with the 'sup' norm however is that it does not satisfy the parallelogram law Eq. (42) which means that it cannot be associated with this or that inner product.

Problem 25. Show by an example that the 'sup' norm does not imply the parallelogram law.

Hence here we have an example of a Banach space which is not a Hilbert space.

Countably infinite orthonormal system. Let $\{|e_j\rangle\}$, $j = 1, 2, 3, \dots$ be an infinite countable orthonormal set of vectors in some infinite-dimensional inner-product vector space. The series

$$\sum_{j=1}^{\infty} \langle e_j|x\rangle \, |e_j\rangle \tag{48}$$

is called Fourier series for the vector $|x\rangle$ with respect to the given orthonormal systems; the numbers $\langle e_j|x\rangle$ are called Fourier coefficients.
Theorem. Partial sums of the Fourier series form a Cauchy sequence.
Proof. We need to show that $\{|x^{(n)}\rangle\}$, where

$$|x^{(n)}\rangle = \sum_{j=1}^{n} \langle e_j|x\rangle \, |e_j\rangle \tag{49}$$

a Cauchy sequence. From (39) we have

$$\|x^{(m)} - x^{(n)}\|^2 = \sum_{j=n}^{m} |\langle e_j|x\rangle|^2 \qquad (m > n) \, , \tag{50}$$

which means that it is sufficient to show that the real-number series

$$\sum_{j=1}^{\infty} |\langle e_j|x\rangle|^2 \tag{51}$$

converges. The series (51) is non-negative, and to prove its convergence it is enough to demonstrate that it is bounded from above. This is done by utilizing the inequality

$$\langle \, x - x^{(n)} | \, x - x^{(n)} \rangle \; \geq \; 0 \, . \tag{52}$$

A straightforward algebra shows that

$$\langle \, x - x^{(n)} | \, x - x^{(n)} \rangle \; = \; \|x\|^2 - \sum_{j=1}^{n} |\langle e_j|x\rangle|^2 \, . \tag{53}$$

Hence, we obtain

$$\sum_{j=1}^{n} |\langle e_j|x\rangle|^2 \; \leq \; \|x\|^2 \qquad \text{(Bessel inequality)} \, , \tag{54}$$

and prove the theorem.

Moreover, rewriting Eq. (53) as

$$\|x - x^{(n)}\| \; = \; \sqrt{\|x\|^2 - \sum_{j=1}^{n} \, |\langle e_j|x\rangle|^2} \; . \tag{55}$$

we see that for the series (49) to *converge* to the vector $|x\rangle$ it is necessary and sufficient to satisfy the condition

$$\sum_{j=1}^{\infty} \, |\langle e_j|x\rangle|^2 = \|x\|^2 \qquad \text{(Parseval relation)} \; . \tag{56}$$

Spanning. As we discussed earlier, a set $\tilde{\nu}$ of all linear combinations constructed out of a finite number of given vectors, $\{|\phi_j\rangle\}$ $(j = 1, 2, \ldots, n)$ forms a subspace of the original vector space ν. In this connection we use the term *span*: $\tilde{\nu}$ is a span of vectors $\{|\phi_j\rangle\}$. Equivalently, we say that vectors $\{\phi_j\}$ span subspace $\tilde{\nu}$.

In a Hilbert space, the notion of span can be generalized to a countably infinite number of vectors in a system. Now we consider all the Cauchy sequences that can be constructed out of *finite-number* linear combinations of the vectors of our system. Without loss of generality, we can consider only orthonormal systems (ONS), and only special Cauchy sequences. Namely, the ones that correspond to Fourier series for elements of the original vector space in terms of the given ONS. The vector properties of the new set follow from the definition of the Fourier series. Denoting by $|x'\rangle$ the Fourier series corresponding to the vector $|x\rangle$, we see that $|x'\rangle + |y'\rangle = |(x + y)'\rangle$ and $\alpha|x'\rangle = |(\alpha x)'\rangle$. The vector $|x'\rangle$ is called *projection* of the vector $|x\rangle$ on the subspace spanned by the orthonormal system.

$\diamond$ In Quantum Mechanical *Measurement Theory*, such projections play most crucial part: They describe the collapse of the wave function after the measurement. For example, the measurement of the *sign* of momentum of 1D particle projects the wavefunction onto subspace spanned by all the eigen states of momentum corresponding to the observed sign.

By the definition of the span $\tilde{\nu}$, corresponding ONS forms an orthonormal basis (ONB) in it, that is any vector $x \in \tilde{\nu}$ can be represented as the Fourier series over this ONS, that converges to x. Correspondingly, if ONS spans the whole Hilbert space, then it forms ONB in this Hilbert space, and for *any* vector of the Hilbert space the Fourier series converges to

Completion. Actually, *any* normed vector space can be upgraded to a *complete* space. And any inner-product vector space can be upgraded to a Hilbert space. Corresponding vector space is called *completion*. The procedure of completing an incomplete vector space ν is as follows. Consider a set $\tilde{\nu}$ of all Cauchy sequences $\{x_k\} \in \nu$. The set $\tilde{\nu}$ is a vector space with respect to the addition and multiplication by a complex number α defined as

$$|\,\{x_k\}\,\rangle \;+\; |\,\{y_k\}\,\rangle \;=\; |\,\{x_k + y_k\}\,\rangle \;, \tag{57}$$

$$\alpha\,|\,\{x_k\}\,\rangle \;=\; |\,\{\alpha\, x_k\}\,\rangle \;. \tag{58}$$

[That is the sum of sequences is defined as the sequence of sums; the product of a sequence and a number is the sequence of products.] We also need to introduce the *equality* as

$$|\,\{x_k\}\,\rangle \;=\; |\,\{y_k\}\,\rangle \;, \qquad \text{if} \qquad \lim_{k\to\infty} \|x_k - y_k\| \;=\; 0 \;. \tag{59}$$

The space $\tilde{\nu}$ naturally contains ν as a subspace, since all convergent in ν sequences can be identified with their limits. The crucial point is to introduce the *norm/inner product* in $\tilde{\nu}$. This is done by an observation (the proof is not difficult, but goes beyond the scope of our course) that for any Cauchy sequence the sequence of norms is convergent, and for any two Cauchy sequences in an inner-product space the sequence of inner products is convergent. One then simply defines

$$\|\{x_k\}\| \;=\; \lim_{k\to\infty} \|x_k\| \;, \tag{60}$$

and, if ν is an inner-product space,

$$\langle\,\{y_k\}\,|\,\{x_k\}\,\rangle \;=\; \lim_{k\to\infty} \langle\, y_k\,|\,x_k\,\rangle \;. \tag{61}$$

Isomorphism of Hilbert spaces with a countably infinite basis. Given a Hilbert space with a countably infinite basis $\{|e_j\rangle\}$, one can construct an isomorphism of this space with a particular space, l_2, which consists of *infinite* complex-number rows, $(x_1, x_2, x_3, \ldots)$, subject to the condition

$$\sum_{j=1}^{\infty} |x_j|^2 < \infty \;. \tag{62}$$

The construction of the isomorphism is absolutely the same as in the previously considered case of the finite-dimensional space, and we do not need to repeat it. The only necessary remark is that the convergence of corresponding series in the space l_2 is guaranteed by Eq. (62). This isomorphism means that all the Hilbert spaces with a countably infinite basis have a similar structure which is very close to that of a finite-dimensional Hilbert space.

Inner product space of functions. Let us try to define an inner product in the space $C[a,b]$ as

$$\langle\, f \,|\, g \,\rangle = \int_a^b \bar{f} g \, w \, \mathrm{d}x \;, \tag{63}$$

where $w \equiv w(x)$ is some real positive-definite function [for the sake of briefness, below we set $w \equiv 1$, since we can always restore it in all the integrals by $\mathrm{d}x \to w(x)\,\mathrm{d}x$.] All the axioms of the inner product are satisfied. What about completeness? The norm now is given by

$$\|f\| = \sqrt{\int_a^b |f|^2 \, \mathrm{d}x} \;, \tag{64}$$

and it is easy to show that the space $C[a,b]$ is not complete with respect to this norm. Indeed, consider a Cauchy sequence of functions $f_k(x)$, $x \in [-1,1]$, where $f_k(x) = 0$ at $x \in [-1,0]$, $f_k(x) = kx$ at $x \in [0,1/k]$, and $f_k(x) = 1$ at $x \in [1/k,1]$. It is easily seen that in the limit of $k \to \infty$ the sequence $\{f_k\}$ converges to the function $f(x)$ such that $f(x) = 0$ at $x \in [-1,0]$, $f(x) = 1$ at $x \in (-1,0]$. But the function $f(x)$ is discontinuous at $x = 0$, that is $f \notin C[a,b]$.

Is it possible to construct a Hilbert space of functions? The answer is "yes", but the theory becomes quite non-trivial. Here we confine ourselves with outlining some most important results. First, one has to understand the integral in the definition of the inner product (63) as a *Lebesgue integral*, which is a more powerful integration scheme than the usual one. Then, one introduces the set of functions, $\mathcal{L}_2[a,b]$, by the following rule: $f \in \mathcal{L}_2[a,b]$ if the Lebesgue integral

$$\int_a^b |f|^2 \, \mathrm{d}x \tag{65}$$

does exist. For the set $\mathcal{L}_2[a,b]$ to be a vector space, one has to introduce a weaker than usual notion of equality. Namely, two functions, $f(x)$ and $g(x)$, if considered as vectors of the space $\mathcal{L}_2[a,b]$, are declared "equal" if

$$\int_a^b |f-g|^2 \,\mathrm{d}x = 0 \; . \tag{66}$$

Fore example, the function $f(x) = 0$ at $x \in [a,b)$ and $f(b) = 1$ is "equal" to the function $g(x) \equiv 0$. It can be proven that this definition of equality implies that the two "equal" functions coincide *almost everywhere*, where the term 'almost everywhere' means that the set of points where the functions do not coincide is of the *measure zero.* [A set of points is of measure zero if it can be covered by a countable set of intervals of arbitrarily small total length.] Correspondingly, the convergence by the inner product norm is actually a convergence *almost everywhere.*

According to the *Riesz-Fisher theorem*, the vector space $\mathcal{L}_2[a,b]$ is a Hilbert space with countably infinite basis.

Also important is the *Stone-Weierstrass theorem* which states that the set of polynomials $\{x^k, k = 0,1,2,\ldots\}$ forms a basis in $\mathcal{L}_2[a,b]$. That is any $f \in \mathcal{L}_2[a,b]$ can be represented as a series

$$|f\rangle = \sum_{k=0}^{\infty} a_k \, x^k \; , \tag{67}$$

which converges to $|f\rangle$ by the inner product norm, that is the series (67) converges to $f(x)$ almost everywhere. The basis of polynomials $\{x^k, k = 0,1,2,\ldots\}$ is not orthogonal. However, one can easily orthonormalize it by the Gram-Schmidt process. The orthonormalized polynomials are called *Legendre polynomials.* Clearly, the particular form of Legendre polynomials depends on the interval $[a,b]$.

It is worth noting that while the space $\mathcal{L}_2[a,b]$ contains pretty weird functions—for example, featuring an infinite number of jumps—the polynomial basis consists of analytic functions. This is because any function in $\mathcal{L}_2[a,b]$ can be approximated to any given accuracy—in the integral sense of Eq. (63)—by some polynomial. In this connection they say that the set of all polynomials is *dense* in $\mathcal{L}_2[a,b]$.