

## Fundamental Solution

Consider the following generic equation:

$$Lu(X) = f(X) . \quad (1)$$

Here  $X = (\mathbf{r}, t)$  is the space-time coordinate (if either space or time coordinate is absent, then  $X \equiv t$ , or  $X \equiv \mathbf{r}$ , respectively);  $L$  is a linear differential operator acting on the unknown function  $u(X)$  and producing some given function  $f(X)$ . The function  $u(X)$  is supposed to be defined everywhere in the time-space. Finally, we assume that the operator  $L$  is invariant with respect to the translations in space and time. Under these conditions, one can look for a generic solution in terms of the Green's function:

$$u(X) = \int G(X - X_0) f(X_0) dX_0 , \quad (2)$$

where  $G$  satisfies the equation

$$LG(X) = \delta(X) . \quad (3)$$

The function  $G(X)$  is called fundamental solution of the operator  $L$ . The Green's function is then obtained by simply replacing  $X \rightarrow X - X_0$ .

Time-dependent problems are often the Cauchy problems when the function  $u$  is defined not only by the operator  $L$ , but also by a particular initial conditions at  $t = 0$ . It is remarkable, however, that the initial conditions can be *absorbed* into the function  $f(X)$ , so that the fundamental solution solves the Cauchy problem as well. The procedure is generic and straightforward. Let us illustrate it with the harmonic oscillator. We have the Cauchy problem

$$\ddot{u} + \omega^2 u = f(t) , \quad (4)$$

$$u(0) = u_0 , \quad \dot{u}(0) = v_0 . \quad (5)$$

Without loss of generality, we assume that  $f(t) \equiv 0$  at  $t < 0$ , because the region  $t < 0$  is undefined for the Cauchy problem. Given the solution  $u(t)$ , we construct a new function,  $\tilde{u}(t)$ , defined for *any*  $t$ , according to the following prescription

$$\tilde{u}(t) = \begin{cases} u(t) , & t \geq 0 , \\ 0 , & t < 0 . \end{cases} \quad (6)$$

Now we note that our  $\tilde{u}(t)$  satisfies Eq. (4) in two regions:  $t < 0$  and  $t > 0$ . But not at  $t = 0$ , because of the jump-like behavior at this point. Applying differential operators to our function in a *generalized* sense, we see that

$$\dot{\tilde{u}}(t) = \begin{cases} \dot{u}(t) + u_0 \delta_+(t), & t \geq 0, \\ 0, & t < 0, \end{cases} \quad (7)$$

and, correspondingly,

$$\ddot{\tilde{u}}(t) = \begin{cases} \ddot{u}(t) + u_0 \delta'_+(t) + v_0 \delta_+(t), & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (8)$$

Here  $\delta'_+(t)$  is the  $\delta$ -function slightly modified in such a way that the following integrals are defined

$$\int_0^\infty \delta_+(t) dt = 1, \quad \int_{-\infty}^0 \delta_+(t) dt = 0. \quad (9)$$

[Below we will need this to remove an ambiguity of integration from  $t = 0$ .] We see that the function  $\tilde{u}$  satisfies (and thus can be found from!) the equation

$$\ddot{\tilde{u}} + \omega^2 \tilde{u} = f(t) + u_0 \delta'_+(t) + v_0 \delta_+(t). \quad (10)$$

This equation has the generic structure of Eq. (1) and we thus have:

$$\tilde{u}(t) = \int G(t - t_0) [f(t_0) + u_0 \delta'_+(t_0) + v_0 \delta_+(t_0)] dt_0. \quad (11)$$

That is

$$\tilde{u}(t) = \int_0^\infty G(t - t_0) f(t_0) dt_0 + v_0 G(t) + u_0 G_t(t). \quad (12)$$

The function  $G$  is found from

$$\ddot{G} + \omega^2 G = \delta_+(t) \quad (13)$$

with the boundary condition

$$G(t) \equiv 0, \quad \text{at } t < 0. \quad (14)$$

This boundary condition is clear from the fact that for *any* function  $f(t)$ , the expression (12) should be identically equal to zero at  $t < 0$ .

The solution to (13)-(14) is easily found by Laplace transform. The answer reads

$$G(t) = \frac{\theta(t)}{\omega} \sin \omega t, \quad (15)$$

$$\theta(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (16)$$

Finally, we obtain

$$\tilde{u}(t) = (1/\omega) \int_0^t \sin[\omega(t-t')] f(t') dt' + (v_0/\omega) \sin \omega t + u_0 \cos \omega t. \quad (17)$$

This technique is easily generalized to other Cauchy problems. One just constructs the function  $\tilde{u}(\mathbf{r}, t)$  in accordance with (6) and substitutes it into the original equation. Because of the discontinuity, the derivatives with respect to time will produce extra terms that should be added to the original function  $f$ . We illustrate this by considering the wave equation [ $u \equiv u(\mathbf{r}, t)$ ]:

$$\square u = f(\mathbf{r}, t), \quad (18)$$

$$\square = \frac{\partial^2}{\partial t^2} - \nabla^2. \quad (19)$$

$$u(\mathbf{r}, 0) = u_0(\mathbf{r}), \quad u_t(\mathbf{r}, 0) = v_0(\mathbf{r}). \quad (20)$$

In accordance with the above-described procedure, we start with expanding the domain of definition of  $t$  to the whole number axis, simultaneously assuming that  $f(\mathbf{r}, t) \equiv 0$  at  $t < 0$ . Then we introduce the function,  $\tilde{u}(\mathbf{r}, t)$ :

$$\tilde{u}(\mathbf{r}, t) = \begin{cases} u(\mathbf{r}, t), & t \geq 0, \\ 0, & t < 0, \end{cases} \quad (21)$$

and calculate its time derivatives to see what is the effect of acting on it with the operator  $\partial^2/\partial t^2$ .

$$\tilde{u}_t(\mathbf{r}, t) = \begin{cases} u_t(\mathbf{r}, t) + u_0(\mathbf{r}) \delta_+(t), & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (22)$$

$$\tilde{u}_{tt}(\mathbf{r}, t) = \begin{cases} u_{tt}(\mathbf{r}, t) + u_0(\mathbf{r}) \delta'_+(t) + v_0(\mathbf{r}) \delta_+(t), & t \geq 0, \\ 0, & t < 0. \end{cases} \quad (23)$$

Acting with the operator  $\square$  on  $\tilde{u}$ , with (23) taken into account, we get

$$\square \tilde{u} = \tilde{f}(\mathbf{r}, t), \quad (24)$$

where

$$\tilde{f}(\mathbf{r}, t) = f(\mathbf{r}, t) + u_0(\mathbf{r}) \delta'_+(t) + v_0(\mathbf{r}) \delta_+(t). \quad (25)$$

The problem (24) is solved by the Green's function method:

$$\tilde{u}(\mathbf{r}, t) = \int G(\mathbf{r} - \mathbf{r}_0, t - t_0) \tilde{f}(\mathbf{r}, t) d\mathbf{r}_0 dt_0 . \quad (26)$$

The Green's function is the fundamental solution of the operator  $\square$ :

$$\square G(\mathbf{r}, t) = \delta_+(t) \delta(\mathbf{r}) , \quad (27)$$

with the condition

$$G(\mathbf{r}, t) \equiv 0 \quad \text{at} \quad t < 0 . \quad (28)$$

As it follows from (25), the final answer for the function  $u(\mathbf{r}, t)$ —in terms of  $G$  and its time derivative—reads

$$\begin{aligned} u(\mathbf{r}, t) = & \int_0^t dt_0 \int d\mathbf{r}_0 G(\mathbf{r} - \mathbf{r}_0, t - t_0) f(\mathbf{r}_0, t_0) + \\ & + \int d\mathbf{r}_0 G(\mathbf{r} - \mathbf{r}_0, t) v_0(\mathbf{r}_0) + \int d\mathbf{r}_0 G_t(\mathbf{r} - \mathbf{r}_0, t) u_0(\mathbf{r}_0) . \end{aligned} \quad (29)$$

The problem of obtaining  $G$  from (27)-(28) can be solved by a combination of Fourier and Laplace transforms—Fourier transform with respect to the coordinates and Laplace transform with respect to time:

$$G(\mathbf{r}, t) = \int_C \frac{dp}{2\pi i} e^{pt} \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\mathbf{r}} g(\mathbf{k}, p) . \quad (30)$$

[Here  $C$  is a proper contour in the complex  $p$ -plane.] Note that (30) automatically implies (28). Plugging (30) into (27) and taking into account that

$$\delta_+(t) \delta(\mathbf{r}) = \int_C \frac{dp}{2\pi i} e^{pt} \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\mathbf{r}} 1 , \quad (31)$$

we find

$$\int_C \frac{dp}{2\pi i} e^{pt} \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\mathbf{r}} (p^2 + k^2) g(\mathbf{k}, p) = \int_C \frac{dp}{2\pi i} e^{pt} \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\mathbf{r}} 1 . \quad (32)$$

Fourier/Laplace transform is *unique*. Hence

$$g(\mathbf{k}, p) = \frac{1}{p^2 + k^2} , \quad (33)$$

and we just need to restore  $G(\mathbf{r}, t)$  from  $g(\mathbf{k}, p)$ . First, we perform inverse Laplace transform:

$$g(\mathbf{k}, t) = \theta(t) \sum_{\text{all poles}} \text{Res} \frac{e^{pt}}{p^2 + k^2} = \theta(t) \frac{\sin kt}{k}. \quad (34)$$

Apart from the  $\theta$ -function which is  $k$ -independent and thus causes no problem when doing inverse Fourier transform, the r.h.s. of (34) is quite familiar for us from the section about the Green's functions of the wave equation. It is what we were calling  $G^{(1)}(\mathbf{r}, t)$  there. Hence, we write down the answers.

$$G(x, t) = \theta(t) [\text{sgn}(t+x) + \text{sgn}(t-x)]/4 \quad (d=1), \quad (35)$$

$$G(\mathbf{r}, t) = \frac{1}{2\pi} \frac{\theta(t-r)}{\sqrt{t^2 - r^2}} \quad (d=2), \quad (36)$$

$$G(\mathbf{r}, t) = \frac{1}{4\pi r} \delta(t-r) \quad (d=3). \quad (37)$$

Note that for  $d=2, 3$  we do not need to write the factor  $\theta(t)$  since the functions are automatically equal to zero at  $t < 0$ .

The 3D case is a very special one: The integration over  $t_0$  in (29) is completely removed by the  $\delta$ -functional form of the Green's function, so that the final answer acquires the famous *retarded-potential* form, where the integration is over the spatial variable only, but with corresponding time-retardation.

**Problem 47.** For each of the Cauchy problems listed below do the following.

(a) By introducing the supplementary function  $\tilde{u}$  in analogy with (6), reformulate the problem in the generic form of Eq. (1) by absorbing the initial condition(s) into the function  $f$ , in analogy with (10).

(b) Write down the solution in terms of the fundamental solution  $G$ . [At this point you are not supposed to find the explicit form of  $G$ .] Pay special attention to the limits of integration with respect to time, following from the condition that  $G(t) \equiv 0$  at  $t < 0$ .

(c) Find the form of  $G$  for the problems (i) and (ii) by *both* Fourier and Laplace transforms.

(d) Find the form of  $G$  for the problem (iii) by combined Fourier-Laplace transform—Fourier with respect to the coordinates and Laplace with respect to time.

List of Cauchy problems:

(i) A simple differential equation ( $u \equiv u(t)$ ,  $\gamma > 0$  is a parameter):

$$\dot{u} + \gamma u = f(t) , \quad (38)$$

$$u(0) = u_0 . \quad (39)$$

(ii) Damped harmonic oscillator ( $u \equiv u(t)$ ,  $\gamma > 0$  is the damping coefficient):

$$\ddot{u} + \gamma \dot{u} + \omega^2 u = f(t) , \quad (40)$$

$$u(0) = u_0 , \quad u_t(0) = v_0 . \quad (41)$$

(iii) Heat/Schrödinger equation: [ $u \equiv u(\mathbf{r}, t)$ ,  $\gamma = 1$  for the heat equation; for the Schrödinger equation  $u$  is complex and  $\gamma = -i$ ]:

$$\gamma u_t - \Delta u = f(\mathbf{r}, t) , \quad (42)$$

$$u(\mathbf{r}, 0) = u_0(\mathbf{r}) . \quad (43)$$