

Laplace Transform

Suppose we have a function $f(t)$ defined for $t \in [0, \infty)$. [In typical cases t is time and $f(t)$ is either a solution for some Cauchy problem with the initial condition at $t = 0$, or some external force.] The function

$$F(p) = \int_0^{\infty} e^{-pt} f(t) dt \quad (1)$$

is called Laplace transform of the Function $f(t)$. The idea behind the Laplace transform is the same as the idea behind the Fourier transform.—It converts derivatives into polynomials and thus is a powerful tool for solving Cauchy problems for linear differential equations. Doing the integral

$$F_n(p) = \int_0^{\infty} e^{-pt} f^{(n)}(t) dt \quad (2)$$

by parts, we find ($f^{(0)} = f$ and $F_0 = F$)

$$F_n(p) = -f^{(n-1)}(0) + pF_{n-1}(p) . \quad (3)$$

That is

$$F_n(p) = -\sum_{j=1}^n p^{(n-j)} f^{(j-1)}(0) + p^n F(p) . \quad (4)$$

In contrast to the Fourier transform, the inverse Laplace transform has to deal with the analytic continuation of the function $F(p)$ into the complex plane of the variable p . Defining

$$p_1 = \operatorname{Re} p , \quad p_2 = \operatorname{Im} p , \quad (5)$$

we have

$$F(p_1 + ip_2) = \int_0^{\infty} e^{-(p_1 + ip_2)t} f(t) dt . \quad (6)$$

Introducing then a new function

$$\tilde{f}(t) = \begin{cases} e^{-p_1 t} f(t) , & t \geq 0 , \\ 0 , & t < 0 , \end{cases} \quad (7)$$

we notice that $F(p_1 + ip_2)$ is nothing else than its Fourier transform:

$$F(p_1 + ip_2) = \int_{-\infty}^{\infty} e^{-ip_2 t} \tilde{f}(t) dt , \quad (8)$$

were p_2 is the variable of the transform and p_1 is a parameter. Normally, the integral (8) converges perfectly for large enough p_1 . For example, this is always the case when $\exists \gamma$ such that

$$|f(t)| < e^{\gamma t}, \quad (9)$$

and it is sufficient to take $p_1 > \gamma$. Below we will be assuming that Eq. (9) holds true. Restoring the function $\tilde{f}(t)$ by the inverse Fourier transform,

$$\tilde{f}(t) = \int_{-\infty}^{\infty} e^{ip_2 t} F(p_1 + ip_2) dp_2 / 2\pi, \quad (10)$$

we ultimately get

$$f(t) = \int_{-\infty}^{\infty} e^{(p_1 + ip_2)t} F(p_1 + ip_2) dp_2 / 2\pi. \quad (11)$$

Calculating the derivative $\partial/\partial p^*$ under the sign of the integral in (6), we see that

$$\frac{\partial F(p)}{\partial p^*} \equiv 0, \quad (p_1 > \gamma). \quad (12)$$

[The condition $p_1 > \gamma$ guaranties the convergence of the integral.—Otherwise differentiating under the sign of integral is ill-defined.] Hence, the function $F(p)$ is analytic at $\text{Re } p > \gamma$, and we can look at the integral (11) as the contour integral

$$f(t) = \frac{1}{2\pi i} \int_{C_0} e^{pt} F(p) dp, \quad (13)$$

where C_0 is a vertical line in the complex- p plane (so that $dp = i dp_2$) corresponding to $\text{Re } p > \gamma$. Normally the contour C_0 can be closed into the closed contour C —see Fig. 1—by the Jordan's lemma.

$$f(t) = \frac{1}{2\pi i} \oint_C e^{pt} F(p) dp, \quad (14)$$

Hence, the contribution to $f(t)$ comes from *all* the poles of the function $e^{pt} F(p)$, since there are no poles at the right-hand side of the contour C_0 .

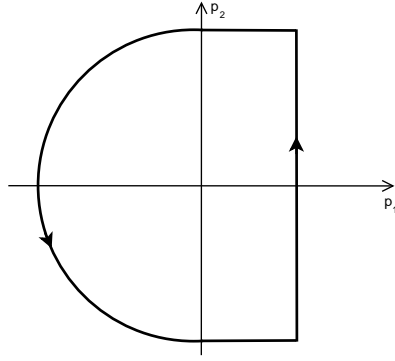


Figure 1: Contour C.

Application to linear differential equations

We illustrate the Laplace transform technique by a simple example of harmonic oscillator with a time-dependent external force $f(t) = t$.

$$\ddot{x} + \omega^2 x = t, \quad (15)$$

$$x(0) = a, \quad \dot{x}(0) = b. \quad (16)$$

The starting point is to Laplace-transform the equation:

$$\int_0^\infty dt e^{-pt} [\ddot{x} + \omega^2 x] = \int_0^\infty dt e^{-pt} t. \quad (17)$$

For the derivatives in the l.h.s. of (17) we use (4), with (16) taken into account. This yields

$$\int_0^\infty dt e^{-pt} [\ddot{x} + \omega^2 x] = (p^2 + \omega^2)X(p) - pa - b, \quad (18)$$

where

$$X(p) = \int_0^\infty dt e^{-pt} x(t). \quad (19)$$

The integral in the r.h.s. of (17) needs to be done explicitly:

$$\int_0^\infty dt e^{-pt} t = -\frac{\partial}{\partial p} \int_0^\infty dt e^{-pt} = -\frac{\partial}{\partial p} \frac{1}{p} = \frac{1}{p^2}. \quad (20)$$

We thus have

$$(p^2 + \omega^2)X(p) - pa - b = \frac{1}{p^2}, \quad (21)$$

and find

$$X(p) = \frac{1 + p^2(pa + b)}{p^2(p^2 + \omega^2)} = \frac{1 + p^2(pa + b)}{p^2(p - i\omega)(p + i\omega)}. \quad (22)$$

Now we perform the inverse transform

$$x(t) = \frac{1}{2\pi i} \int_C \frac{[1 + p^2(pa + b)] e^{pt}}{p^2(p - i\omega)(p + i\omega)} dp . \quad (23)$$

That is we sum up all the residues of the function

$$\frac{[1 + p^2(pa + b)] e^{pt}}{p^2(p - i\omega)(p + i\omega)} . \quad (24)$$

There are three poles. The two simple poles $p = \pm i\omega$ come from the l.h.s. of the original differential equation, while the second-order pole at $p = 0$ is totally due to the particular form of the external force. The residue calculus routine leads to the final answer

$$x(t) = t/\omega^2 + a \cos \omega t + (b/\omega - 1/\omega^3) \sin \omega t . \quad (25)$$

Problem 45. Use the Laplace transform method to solve for the evolution of a damped harmonic oscillator with a time-independent external force:

$$\ddot{x} + \omega^2 x + \gamma \dot{x} = f_0 , \quad (26)$$

$$x(0) = 0 , \quad \dot{x}(0) = 0 , \quad (27)$$

Find $x(t)$; make sure it is real and does satisfy the equation and the initial conditions. Compare two cases: $0 < \gamma < 2\omega$ and $\gamma \geq 2\omega$.

If we are dealing with a system of linear differential equations, the procedure is the same. We Laplace-transform each of the equations and arrive at a system of *algebraic* equations for the Laplace transforms of the unknown functions. From this system we find each of the Laplace transforms and then perform the inverse transforms.

Problem 46. Use the Laplace transform method to solve for the evolution of one of two coupled harmonic oscillators (the coupling force is proportional to the relative displacement of the oscillators, the proportionality coefficient $\lambda > 0$):

$$\ddot{x}_1 + \omega_1^2 x_1 = \lambda(x_2 - x_1) , \quad (28)$$

$$\ddot{x}_2 + \omega_2^2 x_2 = \lambda(x_1 - x_2) . \quad (29)$$

Find $x_1(t)$ for $t > 0$ under the following initial conditions:

$$x_1(0) = 0 , \quad \dot{x}_1(0) = 0 , \quad x_2(0) = 0 , \quad \dot{x}_2(0) = 1 . \quad (30)$$

Make sure $x_1(t)$ is real and does satisfy the initial conditions.