Homework Three Solutions

Keith Fratus

July 26, 2011

1 Problem One

1.1 Part a

The key thing to notice here is that D and M have the same eigenvalues and the same determinant, so if we can show that the product of the eigenvalues is equal to the determinant for D, then it must also be true for M.

To see this, recall that we have, for any two matrices A and B,

$$\det(AB) = \det(A)\det(B), \tag{1}$$

which is given in equation 6.6 in Boas. Therefore,

$$\begin{aligned}\det(D) = \det(C^{-1}MC) = \det(C^{-1})\det(M)\det(C) = \det(C^{-1})\det(C)\det(M) = \\ \det(C^{-1}C)\det(M) = \det(I)\det(M) = \det(M),\end{aligned} \tag{2}$$

and so the determinants are the same.

To see that the eigenvalues are the same, it is enough to show that the two matrices have the same characteristic polynomial. Notice that

$$\begin{aligned}\det(D - \lambda I) = \det(C^{-1}MC - \lambda C^{-1}C) = \det(C^{-1}(M - \lambda I)C) = \\ \det(C^{-1})\det(M - \lambda I)\det(C) = \det(M - \lambda I),\end{aligned} \tag{3}$$

where again the last equality follows from

$$\det(C^{-1})\det(C) = \det(C^{-1}C) = \det(I) = 1 \tag{4}$$

So we see that M and D have the same characteristic polynomial, and thus the same eigenvalues.

As a result of this, if the determinant of D is given by the product of its eigenvalues, then the same must hold for M, since it has the same determinant and same eigenvalues.

Showing this for D is quite trivial. Since D is diagonal, its determinant is just the product of its diagonal entries, since no matter which row or column we expand along, there will only be one non-zero term for each expansion. For example,

$$\det\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 8 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 4\end{pmatrix} = 1 * \det\begin{pmatrix}8 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{pmatrix} = 1 * 8 * \det\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix} = 1 * 8 * 3 * 4. \tag{5}$$

Likewise, the diagonal entries are the eigenvalues of D. To see this, note that D - λI is again diagonal, with λ subtracted from each of the diagonal entries. So the characteristic equation becomes

$$\det(D - \lambda I) = \prod_i (D_{ii} - \lambda) = 0, \tag{6}$$

which is already the factorization of the polynomial, indicating that the diagonal entires of D are the solutions to the eigenvalue problem.

So the determinant of D is just the product of its diagonal entries, which happen to also be the eigenvalues, and so the determinant of D is just the product of its eigenvalues. This then also holds for M.

1.2 Part b

Recall that the trace of a product of matrices is preserved under cyclic permutations. That is, for example,

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB), \tag{7}$$

or,

$$\mathrm{Tr}(QRST) = \mathrm{Tr}(STQR). \tag{8}$$

So, as a result,

$$\mathrm{Tr}(D) = \mathrm{Tr}(C^{-1}MC) = \mathrm{Tr}(CC^{-1}M) = \mathrm{Tr}(M). \tag{9}$$

We already know that the eigenvalues of D and M are the same, and now we know that their traces are the same. So if the trace of D is the sum of its eigenvalues, then this also holds for M.

However, we know that since D is diagonal, its eigenvalues are just its diagonal entries. By definition, the trace is the sum of the diagonal entries, so the trace of D is the sum of its eigenvalues. This therefore also holds for M.

2 Problem Two

The physical set-up of this problem is similar to Figure 12.1 in Boas on page 165, except for the value of the spring constants. Following Boas, I will denote the deviations of the masses from their equilibrium positions as x and y. The calculation of the potential energy of the system proceeds in much the same way as in the example in Boas, aside from the values of the spring constants. The potential energy is now

$$V = \frac{5}{2}kx^2 + k(x-y)^2 + ky^2, \tag{10}$$

since the spring constants are now 5k, 2k, and 2k. The equations of motion then become

$$\begin{aligned} m\ddot{x} &= -\tfrac{\partial V}{\partial x} = -7kx + 2ky \\ m\ddot{y} &= -\tfrac{\partial V}{\partial y} = 2kx - 4ky. \end{aligned} \tag{11}$$

Now, if we assume oscillatory solutions

$$x = x_0 e^{iwt} \ ; \ y = y_0 e^{iwt}, \tag{12}$$

then we find that

$$\begin{aligned} -mw^2 x &= -7kx + 2ky \\ -mw^2 y &= 2kx - 4ky, \end{aligned} \tag{13}$$

which can be written in matrix form as

$$\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{14}$$

where,

$$\lambda \equiv \frac{mw^2}{k}. \tag{15}$$

So we have reduced the problem of finding w to the problem of finding the eigenvalues of the above matrix. I'll leave this eigenvalue calculation to you guys, and simply state that the two eigenvalues can be found to be

$$\lambda_1 = 3, \lambda_2 = 8, \tag{16}$$

so that the frequencies become

$$w_1 = \sqrt{\frac{3k}{m}}, \lambda_2 = \sqrt{\frac{8k}{m}}. \tag{17}$$

The actual motion of the two masses for these frequencies can be found by finding the corresponding eigenvectors. Any general solution to this system can be written as a linear combination of these two solutions.

The physical system we have considered is of course a very idealized one, which is why we were able to write the system of equations (11) in such a simple form. One may wonder if this technique has any applicability for more general systems. The answer is that it most certainly does. The reason is that for any system for which we can deduce the equations of motion, it is (almost) always possible to Taylor expand the terms showing up in (11), for when we expect to have small deviations from equilibrium. Thus, we can use this method to find the characteristic small oscillations for any system, which are referred to as the *normal modes* of the system. If we use the Lagrangian formalism to find the equations of motion, the coordinates don't even need to be spatial coordinates; they could be totally general degrees of freedom. For those interested, Boris Svistunov's notes on small oscillations talk about this subject in more depth.

3 Problem Three

3.1 Part a

As always, we find the eigenvalues according to

$$\det(M - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0, \tag{18}$$

which implies

$$\lambda = \pm 1. \tag{19}$$

To find the eigenvector corresponding to $\lambda = +1$, we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} a \\ b, \end{pmatrix} \tag{20}$$

which means we can take the normalized eigenvector to be

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{21}$$

In a similar fashion, we can find the normalized eigenvector corresponding to λ = -1 as

$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{22}$$

3.2 Part b

As derived by Eli in class, we know that we can construct the Green's function as

$$\hat{G} = \sum_{i=1}^{N} \frac{|\lambda_i\rangle\langle\lambda_i|}{\lambda_i - \lambda}, \tag{23}$$

so long as λ is not equal to one of the eigenvalues. In our case, this becomes

$$\hat{G} = \frac{|\lambda_1\rangle\langle\lambda_1|}{1-\lambda} - \frac{|\lambda_2\rangle\langle\lambda_2|}{1+\lambda}, \tag{24}$$

so that the solution reads

$$|a\rangle = \hat{G}|f\rangle = \frac{|\lambda_1\rangle\langle\lambda_1|f\rangle}{1-\lambda} - \frac{|\lambda_2\rangle\langle\lambda_2|f\rangle}{1+\lambda}. \tag{25}$$

Now, we have

$$\langle\lambda_1|f\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \frac{3}{\sqrt{2}}, \tag{26}$$

along with

$$\langle\lambda_2|f\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \frac{-1}{\sqrt{2}}. \tag{27}$$

Thus, we find that

$$|a\rangle = \frac{1}{2}\left(\frac{3}{1-\lambda}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{1+\lambda}\begin{pmatrix}1\\-1\end{pmatrix}\right), \tag{28}$$

which can be further simplified if one so chooses. If you have any questions as to where these formulas come from, please feel free to ask me questions regarding the subject.

4 Problem Four

I'll address this problem by giving my advice on where to read about these subjects. For parts a, b, and e, my personal advice would be to get a copy of "Introduction to Quantum Mechanics" by David Griffiths, second edition. I would also recommend "Modern Quantum Mechanics" by J. J. Sakurai.

For parts c and d, I don't know that I would suggest any one book in particular, so I would personally recommend doing a Google Books search for these subjects.

Feel free to ask me questions about these subjects if you would like to know more.

5 Problem Five

We assume that our solution has a power series expansion given by

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{29}$$

From this it follows that

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \tag{30}$$

along with

$$y''(x) = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \tag{31}$$

By using k = n-2, we can re-index the previous sum as

$$y''(x) = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k. \tag{32}$$

For now we will not re-index the sum for the first derivative, the reason for which will become obvious in a moment.

Inserting these results into our original differential equation, we have

$$\left(\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k \right) + x \left(\sum_{n=0}^{\infty} n a_n x^{n-1} \right) + \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0. \tag{33}$$

However, since k is just a dummy variable, we can just rename it back to n and bring the sums together, to get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x n a_n x^{n-1} + a_n x^n = 0, \tag{34}$$

which simplifies to

$$\sum_{n=0}^{\infty} [\ (n+2)(n+1) a_{n+2} + (n+1) a_n\ ]\, x^n = 0. \tag{35}$$

For this equation to be satisfied identically, we must have the coefficient on each power of x go to zero. In other words, we must have

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0, \tag{36}$$

which yields

$$a_{n+2} = \frac{-a_n}{(n+2)}. \tag{37}$$

Now, my claim is that the above result tells us

$$a_{2k} = \frac{(-1)^k}{(2k)!!}\ a_0\ ;\ a_{2k+1} = \frac{(-1)^k}{(2k+1)!!}\ a_1, \tag{38}$$

where the double factorial is given by

$$n!! \equiv n(n-2)(n-4)...n_0, \tag{39}$$

where n_0 is equal to 2 if n is even, and 1 if n is odd. If n is equal to zero, then the double factorial is just defined to be 1. To check this, first notice that it holds correctly for the k = 0 cases. To check the expansion for the odd terms, notice that

$$a_{2k+3} = a_{2(k+1)+1} = \frac{(-1)^{k+1}}{(2k+3)!!}\ a_1 = \frac{-(-1)^k}{(2k+3)(2k+1)!!}\ a_1 = \frac{-1}{(2k+3)} a_{2k+1}, \tag{40}$$

and with n = 2k+1, this result reads

$$a_{2k+1+2} = a_{n+2} = \frac{-1}{(n+2)} a_n, \tag{41}$$

which is indeed the correct recursion relation. The formula for the even terms can be verified in a similar manner.

Realizing what the formula should be is somewhat of an art, and involves being able to recognize common patterns when they arise. For example, for the odd terms, if we start doing explicit computations, we see that

$$a_1 = a_1 \ ; \ a_3 = -\frac{1}{3}a_1 \ ; \ a_5 = \frac{1}{5} \cdot \frac{1}{3}a_1, \tag{42}$$

and so we start to realize there is a double factorial pattern emerging, with alternating signs. The same idea occurs with the even terms. However, once we realize what the pattern should be, verifying it is relatively straight-forward using induction, as I did above.

Now that we know the coefficients for the even and odd terms, we can use them to write our final result. If we split the summation into even and odd terms, we have

$$y(x) = \sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1}, \tag{43}$$

which allows us to write

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!!}x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!!}x^{2k+1}. \tag{44}$$

In order to find a_0 and a_1, we would need to know some initial conditions. Notice that

$$y(0) = a_0 \ ; \ y'(0) = a_1 \tag{45}$$

(which you should verify for yourself), so that we can alternatively write

$$y(x) = y(0) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!!}x^{2k} + y'(0) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!!}x^{2k+1}, \tag{46}$$

which gives the full solution in terms of initial conditions.

As an aside, the above result can also be written as

$$y(x) = y(0)e^{-\frac{x^2}{2}} + \sqrt{2}y'(0)F\left(\frac{x}{\sqrt{2}}\right), \tag{47}$$

where the function F(x) is the Dawson Integral, something which I'll let you look up on your own. Personally, I found these explicit summations by looking them up with Mathematica. So the even portion has a summation which comes out to a nice looking function, but the same is not quite true of the odd portion. Notice that if the initial value of the derivative is zero, then we only have the even portion, and the final result is a Gaussian.

6 Problem Six

6.1 Part a

We assume we can write our solution as

$$y(x) = \sum_{l=0}^{\infty} a_l x^l, \tag{48}$$

from which it follows

$$y'(x) = \sum_{l=0}^{\infty} l a_l x^{l-1}, \tag{49}$$

along with

$$y''(x) = \sum_{l=0}^{\infty} l(l-1) a_l x^{l-2}. \tag{50}$$

Notice that we have

$$xy'(x) = \sum_{l=0}^{\infty} l a_l x^l, \tag{51}$$

along with

$$x^2 y''(x) = \sum_{l=0}^{\infty} l(l-1) a_l x^l. \tag{52}$$

If we plug these results back into our original differential equation, we find that we have

$$\left(\sum_{l=0}^{\infty} l(l-1) a_l x^{l-2}\right) - \left(\sum_{l=0}^{\infty} l(l-1) a_l x^l\right) - \left(\sum_{l=0}^{\infty} l a_l x^l\right) + \left(\sum_{l=0}^{\infty} n^2 a_l x^l\right) = 0, \tag{53}$$

where I brought the n^2 term into the last summation. If we re-index the first sum, we can write this as

$$\left(\sum_{l=0}^{\infty} (l+2)(l+1) a_{l+2} x^l\right) - \left(\sum_{l=0}^{\infty} l(l-1) a_l x^l\right) - \left(\sum_{l=0}^{\infty} l a_l x^l\right) + \left(\sum_{l=0}^{\infty} n^2 a_l x^l\right) = 0, \tag{54}$$

and so we can bring the summation all together to get

$$\sum_{l=0}^{\infty} \left[(l+2)(l+1) a_{l+2} - l(l-1) a_l - l a_l + n^2 a_l\right] x^l = 0, \tag{55}$$

or,

$$\sum_{l=0}^{\infty} \left[(l+2)(l+1) a_{l+2} + (n^2 - l^2) a_l\right] x^l = 0. \tag{56}$$

Again, since each coefficient must be identically zero, this implies

$$a_{l+2} = -\frac{(n^2 - l^2)}{(l+2)(l+1)} a_l. \tag{57}$$

In some sense, our problem is now solved. The above formula allows us to generate any arbitrary coefficient in terms of a_0 and a_1, and these two numbers can be found from initial conditions. I'll leave it as an exercise for you guys to see if there's a "pretty" way to write out the solution, instead of leaving it in terms of the above implicit recursion relation.

6.2 Part b

In order to have polynomial solutions, the series expansion needs to terminate at some point. Notice that if we choose n to be an integer, then when n = l, we have

$$a_{l+2} = -\frac{(n^2 - l^2)}{(l+2)(l+1)} a_l = -\frac{(n^2 - n^2)}{(n+2)(n+1)} a_l = 0, \tag{58}$$

which will terminate one of the two recursion chains (keep in mind that since we have a formula for a_{l+2} in terms of a_l, then there are two separate chains, one based on a_0 and the other based on a_1). In order to have the other chain terminate, we need to impose that either a_0 or a_1 is equal to zero, depending on whether or not n is even or odd. This will make sure that one chain terminates, while the other one never begins in the first place, guaranteeing that we get a finite polynomial result.

For T_0, with n = 0, we choose $a_1 = 0$, in order to simply get

$$T_0 = a_0, \tag{59}$$

since all higher order terms will vanish. If we want to impose that $T_0(1) = 1$, then we simply choose

$$T_0 = 1. \tag{60}$$

For T_1, we choose $a_0 = 0$, and thus we find

$$T_1 = a_1 x. \tag{61}$$

Again, imposing that $T_1(1) = 1$, we have simply

$$T_1 = x. \tag{62}$$

For T_2, we choose $a_1 = 0$, and we find that

$$a_2 = -\frac{2^2 - 0^2}{(0+2)(0+1)} a_0 = -2a_0, \tag{63}$$

and so we have

$$T_2(x) = a_0(1 - 2x^2). \tag{64}$$

To achieve the proper boundary conditions, we choose

$$T_2(x) = 2x^2 - 1, \tag{65}$$

which indeed respects $T_2(1) = 1$.

Lastly, for T_3, we choose $a_0 = 0$, and we have

$$a_3 = -\frac{3^2 - 1^2}{(1+2)(1+1)} a_1 = -\frac{4}{3} a_1, \tag{66}$$

which leads to

$$T_3(x) = a_1(x - \frac{4}{3}x^3). \tag{67}$$

In order to respect the boundary conditions, we choose a_1 so that

$$T_3(x) = 4x^3 - 3x. \tag{68}$$

6.3 Part c

As Eli derived in class, we know that if we have an operator

$$\hat{L} = \alpha(x)\frac{\partial^2}{\partial x^2} + \beta(x)\frac{\partial}{\partial x} + \gamma(x), \tag{69}$$

then a suitable weighting function is given by

$$w(x) = \frac{1}{\alpha(x)} \exp\left(\int \frac{\beta(x)}{\alpha(x)}\ dx\right). \tag{70}$$

In our case, we can identify

$$\alpha(x) = 1 - x^2\ ;\ \beta(x) = -x, \tag{71}$$

so that we have

$$\int \frac{\beta(x)}{\alpha(x)}\ dx = \int \frac{x}{x^2 - 1}\ dx = \frac{1}{2}\ln(1 - x^2) = \ln((1 - x^2)^{\frac{1}{2}}), \tag{72}$$

where we have chosen the constant of integration to be zero, since this constant of integration leads to an overall multiplicative constant on the inner product. Thus we have

$$\begin{gathered} w(x) = \tfrac{1}{\alpha(x)} \exp\left(\int \tfrac{\beta(x)}{\alpha(x)}\ dx\right) = \tfrac{1}{1-x^2} \exp(\ln((1 - x^2)^{\frac{1}{2}})) = \\ \tfrac{\sqrt{1-x^2}}{1-x^2} = \tfrac{1}{\sqrt{1-x^2}}. \end{gathered} \tag{73}$$

7 Problem Seven

7.1 Part a

Recall the the Legendre polynomials are either even or odd, depending on whether or not the integer describing them is even or odd. For example, P_0 is even while P_5 is odd. There are a number of ways to see this. One of them is to see that the Rodrigues formula,

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x^2 - 1)^n], \tag{74}$$

involves the n^{th} derivative of an even function. Remembering that the derivative of an even function is odd, and vice versa, we see that indeed, the Legendre polynomials must be even or odd, depending on whether n is even or odd.

We can summarize this statement with the notation

$$P_n(-x) = (-1)^n P_n(x), \tag{75}$$

from which it follows

$$P_n(-1) = (-1)^n P_n(1). \tag{76}$$

However, the Legendre polynomials by definition are normalized such that

$$P_n(1) = 1, \tag{77}$$

so that we arrive at

$$P_n(-1) = (-1)^n. \tag{78}$$

7.2 Part b

Please see Figure 1.

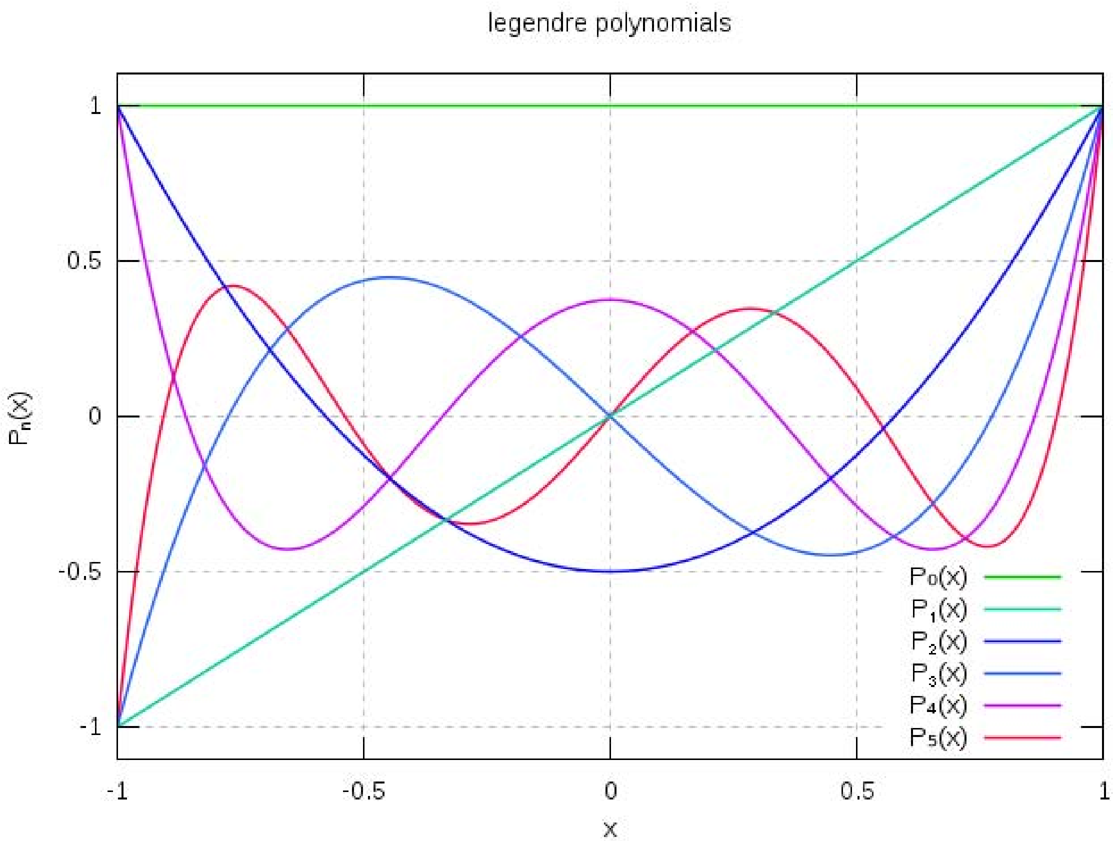

Figure 1: The first six Legendre polynomials, courtesy of Wikipedia