Midterm Solutions

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1 Problem One

1.1 Part a

First, let us suppose that our Hermitian operator is denoted $\hat{H}$, so that the eigenvalue/eigenvector equation reads

$$\hat{H}|a\rangle = \lambda|a\rangle. \tag{1}$$

Now, we can compute

$$\langle a|\hat{H}|a\rangle = \langle a|\left(\hat{H}|a\rangle\right) = \langle a|\left(\lambda|a\rangle\right) = \lambda\langle a|a\rangle, \tag{2}$$

which is a straight-forward application of the eigenvalue/eigenvector equation. Now, we have

$$\langle a|\hat{H}^\dagger|a\rangle = \left(\langle a|\hat{H}|a\rangle\right)^*. \tag{3}$$

This follows from the fact that, from the definition of the adjoint, there is the dual correspondence

$$\langle a|\hat{H}^\dagger \leftrightarrow \hat{H}|a\rangle, \tag{4}$$

along with the fact that swapping the order of an inner product results in a complex conjugation. However, because $\hat{H}$ is Hermitian, it is equal to its adjoint, so we have also

$$\langle a|\hat{H}^\dagger|a\rangle = \langle a|\hat{H}|a\rangle \tag{5}$$

Combining these two results, we find that

$$\langle a|\hat{H}|a\rangle = \left(\langle a|\hat{H}|a\rangle\right)^* \Rightarrow \lambda\langle a|a\rangle = \left(\lambda\langle a|a\rangle\right)^* = \lambda^*\langle a|a\rangle^* \tag{6}$$

Now, because $\langle a|a\rangle$ is a real number, this simplifies to

$$\lambda\langle a|a\rangle = \lambda^*\langle a|a\rangle. \tag{7}$$

Because $\langle a|a\rangle$ must also be strictly positive, we can cancel it from both sides, to arrive at

$$\lambda = \lambda^*, \tag{8}$$

which implies that λ is real. Because no assumptions were made on λ, other than the fact that it was an eigenvalue of $\hat{H}$, we thus see that all eigenvalues of a Hermitian operator must be real.

1.2 Part b

Suppose that we have

$$\hat{H}|a_1\rangle = \lambda_1|a_1\rangle \ ; \ \hat{H}|a_2\rangle = \lambda_2|a_2\rangle, \tag{9}$$

where λ_1 and λ_2 are not equal. We can compute

$$\langle a_1|\hat{H}|a_2\rangle^* = \left(\langle a_1|\left(\hat{H}|a_2\rangle\right)\right)^* = \left(\lambda_2\langle a_1|a_2\rangle\right)^* = \lambda_2\langle a_2|a_1\rangle, \tag{10}$$

where in the last line we got rid of the complex conjugation by swapping the order of the inner product, and also used the fact that λ_2 must be real, since it is the eigenvalue of a Hermitian operator.

Now, we also have

$$\langle a_1|\hat{H}|a_2\rangle^* = \langle a_2|\hat{H}^\dagger|a_1\rangle, \tag{11}$$

which again follows from the dual correspondence

$$\hat{H}|a_2\rangle \leftrightarrow \langle a_2|\hat{H}^\dagger. \tag{12}$$

However, since $\hat{H}$ is Hermitian, we have

$$\langle a_2|\hat{H}^\dagger|a_1\rangle = \langle a_2|\hat{H}|a_1\rangle = \lambda_1\langle a_2|a_1\rangle \tag{13}$$

By comparing these two different results for the value of $\langle a_1|\hat{H}|a_2\rangle^*$, we see that we have

$$\lambda_2\langle a_2|a_1\rangle = \lambda_1\langle a_2|a_1\rangle \Rightarrow (\lambda_2 - \lambda_1)\langle a_2|a_1\rangle = 0. \tag{14}$$

Now, since the eigenvalues are distinct, $(\lambda_2 - \lambda_1)$ is not equal to zero, and so we can cancel it to arrive at

$$\langle a_2|a_1\rangle = 0, \tag{15}$$

which is precisely the statement that the two eigenvectors are orthogonal.

2 Problem Two

2.1 Part a

I'll derive this for σ_x, and then leave the remaining two up to you.

We have the usual formula for matrix multiplication

$$(AB)_{ij} = \sum_k A_{ik}B_{kj}, \tag{16}$$

which, in our present case, implies

$$\left((\sigma_x)^2\right)_{ij} = \sum_k (\sigma_x)_{ik}(\sigma_x)_{kj}. \tag{17}$$

For example, we have

$$\left((\sigma_x)^2\right)_{11} = (\sigma_x)_{11}(\sigma_x)_{11} + (\sigma_x)_{12}(\sigma_x)_{21} = 0\cdot 0 + 1\cdot 1 = 1, \tag{18}$$

as it should be. Proceeding in this way, we also find that

$$\left((\sigma_x)^2\right)_{12} = \left((\sigma_x)^2\right)_{21} = 0\ ;\ \left((\sigma_x)^2\right)_{22} = 1, \tag{19}$$

which implies that the square of σ_x is indeed the identity.

2.2 Part b

We have

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x. \tag{20}$$

Again, we can use the usual formula for the multiplication of matrices. If we use this formula, what we find is that

$$\sigma_x\sigma_y = i\sigma_z \ ; \ \sigma_y\sigma_x = -i\sigma_z. \tag{21}$$

Please feel free to speak with me if you have any confusion as to how to use the matrix multiplication formula to arrive at this result. We see now that

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z. \tag{22}$$

Proceeding in a similar manner, we find that

$$[\sigma_y, \sigma_z] = 2i\sigma_x \ ; \ [\sigma_z, \sigma_x] = 2i\sigma_y. \tag{23}$$

2.3 Part c

A matrix is Hermitian if it is true that

$$A_{ij} = A^*_{ji}, \tag{24}$$

and so to verify that the Pauli matrices are Hermitian, we must verify that this equation holds true for all four components of each Pauli matrix. For σ_y, notice that we have

$$A_{11} = 0 = A^*_{11} \ ; \ A_{12} = -i = A^*_{21} \ ; \ A_{22} = 0 = A^*_{22}, \tag{25}$$

and the fourth equality follows from taking the conjugate of the second equality. Thus, since this equation holds for all of the components, σ_y is Hermitian. Verifying that the other two matrices are Hermitian proceeds in exactly the same way.

3 Problem Three

3.1 Part a

The convention that we have in this problem is that the operator $\hat{R}$ is defined by

$$\hat{R}|1\rangle = |1'\rangle \ ; \ \hat{R}|2\rangle = |2'\rangle, \tag{26}$$

which defines a map that we can represent in the original basis. This convention views $\hat{R}$ as an operator which changes vectors themselves, since we have, for any scalars a and b,

$$\hat{R}\left(\ a|1\rangle + b|2\rangle \ \right) = a|1'\rangle + b|2'\rangle. \tag{27}$$

To represent this operation as a matrix in the original basis, we first realize that the action of our map is

$$\hat{R}|1\rangle = |1\rangle + |2\rangle \ ; \ \hat{R}|2\rangle = |1\rangle + i|2\rangle. \tag{28}$$

If we make the representation

$$|1\rangle \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ; \ |2\rangle \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{29}$$

then we have

$$|1\rangle + |2\rangle \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ ; \ |1\rangle + i|2\rangle \mapsto \begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{30}$$

If R is the matrix which represents the operator $\hat{R}$, then we know that the first column of R is given by the coordinates of the action of $\hat{R}$ on the first basis vector, while the second column of R is given by the coordinates of the action of $\hat{R}$ on the second basis vector (please see my notes from the second discussion section if you are not familiar with why this is the case). Thus, we have

$$\hat{R} \mapsto R = \begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}. \tag{31}$$

To find the inverse, we could use the method of cofactors given in Boas, or simply remember the simple formula for a 2 by 2 matrix which says that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{32}$$

Using this equation for R, we find that

$$R^{-1} = \frac{1}{i-1} \begin{pmatrix} i & -1 \\ -1 & 1 \end{pmatrix} = \frac{(-i-1)}{(i-1)(-i-1)} \begin{pmatrix} i & -1 \\ -1 & 1 \end{pmatrix} = \frac{(1+i)}{2} \begin{pmatrix} -i & 1 \\ 1 & -1 \end{pmatrix}. \tag{33}$$

3.2 Part b

For R to be a unitary matrix, we must have

$$R^{-1} = R^\dagger. \tag{34}$$

Now, the conjugate transpose of R is found by swapping the rows and columns of R, and then conjugating every element, so we see that

$$R^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & -i \end{pmatrix}, \tag{35}$$

which is most certainly not equal to the inverse of R (which we found in part a). Thus, R is *not* a unitary matrix.

3.3 Part c

If we take the column of numbers which represents the coordinates of a vector in the old basis, and apply the inverse of R to is, we should get the coordinates of the vector in the new basis (that is, in this context we are thinking of the vector being fixed, while the coordinates change). First, to test that our matrix inverse is correct (which was not required on the exam, but which I'll do here for clarity), let's try it for one of the original basis vectors. By scaling and subtracting the two equations which define $|1'\rangle$ and $|2'\rangle$, we can perform some algebraic manipulation to find that

$$|1\rangle = \frac{1}{i-1} \left(\ i|1'\rangle - |2'\rangle \ \right), \tag{36}$$

which means that we can represent $|1\rangle$ in the new basis of $\{\ |1'\rangle, |2'\rangle\ \}$ as

$$|1\rangle \mapsto \frac{1}{i-1}\begin{pmatrix} i \\ -1 \end{pmatrix}, \tag{37}$$

whereas in the old basis we had

$$|1\rangle \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{38}$$

Now, we see that

$$R^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{i-1}\begin{pmatrix} i & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{i-1}\begin{pmatrix} i \\ -1 \end{pmatrix}, \tag{39}$$

and so the inverse does indeed send the old coordinates to the new coordinates. In the same way, we can verify that this inverse does the same thing for the vector $|2\rangle$.

Now that we've tested for ourselves that this works, let's apply it to the vector $|x\rangle$. Acting the inverse on the old coordinates, we get

$$R^{-1}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{i-1}\begin{pmatrix} i & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{i-1}\begin{pmatrix} 1+i \\ -2 \end{pmatrix} = \begin{pmatrix} -i \\ 1+i \end{pmatrix}, \tag{40}$$

so that we can write

$$|x\rangle = -i|1'\rangle + (1+i)|2'\rangle. \tag{41}$$

3.4 Part d

We have the formula

$$\sigma_y' = R^{-1}\sigma_y R, \tag{42}$$

which was derived in lecture. Keep in mind that if σ_y is the representation of the operator in the original basis, then σ_y' is the representation of the same operator, except in the new basis. Therefore, this problem reduces to a simple exercise in matrix multiplication. The result that we find is

$$\sigma_y' = \begin{pmatrix} -1 & 0 \\ 1-i & 1 \end{pmatrix}. \tag{43}$$

Please speak to me if you need any help performing these matrix multiplications.

3.5 Part d

The transpose conjugate of σ_y' is found to be

$$(\sigma_y')^\dagger = \begin{pmatrix} -1 & 1+i \\ 0 & 1 \end{pmatrix}, \tag{44}$$

which is clearly not the same as σ_y'. Thus, σ_y' is not Hermitian.

Now, notice also that

$$\langle 1'|1'\rangle = (\ \langle 1| + \langle 2|\ )\cdot(\ |1\rangle + |2\rangle\ ) = \langle 1|1\rangle + \langle 1|2\rangle + \langle 2|1\rangle + \langle 2|2\rangle = 1 + 1 = 2, \tag{45}$$

where in the second-to-last equality I used the fact that the original basis is an orthonormal one. Thus, we have

$$|| \, |1'\rangle \, || = \sqrt{\langle 1'|1'\rangle} = \sqrt{2} \neq 1, \tag{46}$$

and so the new basis cannot be an orthonormal one.

Both of these facts are a result of R not being a unitary transformation.

4 Problem Four

Our operator $\hat{G}$ is defined by

$$\hat{G}|f\rangle = \int_{-\infty}^{\infty} \frac{f(y)}{|x-y|} dy, \tag{47}$$

and our inner product is defined by

$$\langle g|f\rangle = \int_{-\infty}^{\infty} g^*(x) f(x) dx. \tag{48}$$

Therefore, we have

$$\langle g|\hat{G}|f\rangle = \int_{-\infty}^{\infty} g^*(x) \left(\int_{-\infty}^{\infty} \frac{f(y)}{|x-y|} dy \right) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g^*(x) f(y)}{|x-y|} dy dx, \tag{49}$$

where we can pull g*(x) into the second integration, since it is just a constant multiple with respect to the y-integration (note that in my notation, I am using the variable y, instead of x'). Now, notice that

$$\left(\frac{g^*(x) f(y)}{|x-y|} \right)^* = \frac{g(x) f(y)^*}{|x-y|}, \tag{50}$$

where the denominator is unchanged because the absolute value of a real number is another real number. So we can write

$$\langle g|\hat{G}|f\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{g(x) f(y)^*}{|x-y|} \right)^* dy dx = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(x) f(y)^*}{|x-y|} dy dx \right)^*, \tag{51}$$

where in the last step we pull the conjugate out of the integral (which I told you to assume you could do). Now, because x and y are just dummy variables, I can swap them in the double integral, to get

$$\langle g|\hat{G}|f\rangle = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{g(y) f^*(x)}{|y-x|} dx dy \right)^*. \tag{52}$$

Now, because we have

$$g(y) f^*(x) = f^*(x) g(y), \tag{53}$$

along with

$$|y-x| = |x-y|, \tag{54}$$

then we can write

$$\langle g|\hat{G}|f\rangle = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f^*(x) g(y)}{|x-y|} dx dy \right)^*. \tag{55}$$

Lastly, we can reverse the order of integration, to arrive at

$$\langle g|\hat{G}|f\rangle = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f^*(x)g(y)}{|x-y|}dydx\right)^*. \tag{56}$$

By directly comparing with (49), we see that we have the roles of f and g reversed, along with an overall conjugation, so that we have

$$\langle g|\hat{G}|f\rangle = \langle f|\hat{G}|g\rangle^*. \tag{57}$$

Thus, we arrive at the result that $\hat{G}$ is indeed Hermitian.