UNIVERSITY OF CALIFORNIA, SANTA BARBARA
Department of Physics

Phys 100A - Summer 2011
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Midterm Exam

July 7, 2011

Instructions:

- This is a closed notes, closed book exam.
- No calculators, laptops, phones, etc. are allowed.
- Each of 4 problems will be worth 25 points for a total of 100 points, worth 20 percent of your grade.
- Partial credit will be given for attempted solutions displaying competence of the material.
- No collaboration with other students is permitted.
- You have 80 minutes to complete the exam.

Good luck!

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1. Please prove *one* of the following two theorems about Hermitian operators.

(a) Theorem: Eigenvalues of a Hermitian operator are real.
Proof:

(b) Theorem: Eigenvectors corresponding to different eigenvalues of a Hermitian operator are orthogonal.
Proof:

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2. The Pauli spin matrices, which represent electron "spin" in quantum mechanics, are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that $(\sigma_x)^2 = (\sigma_y)^2 = (\sigma_z)^2 = I$, where I is the 2×2 identity matrix.

(b) Recall that the commutator of two operators, A and B, is the operator $[A, B] = AB - BA$. Compute the commutator for each pair of spin matrices in cyclic order; that is, compute $[\sigma_x, \sigma_y]$, $[\sigma_y, \sigma_z]$, and $[\sigma_z, \sigma_x]$.

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2 cont.
(c) Show that the Pauli spin matrices are each Hermitian.

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3. Let $\{|1\rangle, |2\rangle\}$ be an orthonormal basis of the two-dimensional vector space V and let $\{|1'\rangle, |2'\rangle\}$ be a new basis of V, where

$$|1'\rangle = |1\rangle + |2\rangle$$

$$|2'\rangle = |1\rangle + i|2\rangle$$

(a) Write down the matrix R, representing the operator transforming vectors from the old basis to the new basis, and find R^{-1}.

(b) Is R a unitary matrix?

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3 cont.
(c) Consider a vector $|x\rangle \in V$, expressed in terms of the old basis as $|x\rangle = |1\rangle - |2\rangle$. The components of $|x\rangle$ can be represented by a column vector α:

$$|x\rangle \longmapsto \alpha = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Use R^{-1} from part (a) to express $|x\rangle$ in erms of the new basis; that is, find α':

$$|x\rangle \longmapsto \alpha' = \begin{pmatrix} \alpha'_1 \\ \alpha'_2 \end{pmatrix}.$$

and write down $|x\rangle = \alpha'_1|1'\rangle + \alpha'_2|2'\rangle$.

(d) Consider the Hermitian spin matrix $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ from problem 2, representing the operator $\hat{\sigma_y}$ in the old basis. Using R and R^{-1} find σ'_y, the matrix representing $\hat{\sigma'_y}$ in the new basis.

(e) Is σ'_y Hermitian? Is $\{|1'\rangle, |2'\rangle\}$ an orthonormal basis of V? Why or why not?

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4. The "Green's function" operator used in electrostatics is given by the following:

$$\hat{G} = \int_{-\infty}^{\infty} dx' \frac{1}{|x - x'|}$$

Action on $|f\rangle \mapsto f(x) \in V$ thus looks like

$$\hat{G}|f\rangle = \int_{-\infty}^{\infty} dx' \frac{f(x')}{|x - x'|}$$

and taking an inner product using $\hat{G}$ with some $|g\rangle \mapsto g(x) \in V$ is done by

$$\langle g|\hat{G}|f\rangle = \int_{-\infty}^{\infty} dx g^*(x)\hat{G}f(x)$$

Show that $\hat{G}$ is Hermitian by showing

$$\langle g|\hat{G}|f\rangle^* = \langle f|\hat{G}|g\rangle$$

(Hint: The inner product will be a double integral over both dx and dx' and exchanging these variables $x \leftrightarrow x'$will be useful at some point. You need not evaluate any of the integrals.)