# Physics 20 Homework 1
# SIMS 2016

Due: Wednesday, August 17$^{\text{th}}$

1. Imagine a simplified model of human nutrition, where the world population is sustained by only eating corn. Estimate how many ears of corn are being consumed per second on Earth.

   Hint: Yes, this is a serious question.

   Second Hint: Your answer doesn't need to be exact.

2. Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, show that their sum $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ satisfies the law of cosines:

   $$c^2 = a^2 + b^2 + 2ab\cos\theta$$

   where $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ (Hint: remember that by definition, $c^2 \equiv \boldsymbol{c} \cdot \boldsymbol{c}$). Using this result, explain why the triangle inequality

   $$|\boldsymbol{a} + \boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$$

   is true.

3. Consider two vectors whose components depend on time in a given coordinate system,

   $$\boldsymbol{v}(t) = (v_x(t), v_y(t), v_z(t)) \;\; ; \;\; \boldsymbol{w}(t) = (w_x(t), w_y(t), w_z(t))$$

   Prove that the dot product between these two vectors satisfies the identity

   $$\frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{w}) = \boldsymbol{v} \cdot \left(\frac{d\boldsymbol{w}}{dt}\right) + \left(\frac{d\boldsymbol{v}}{dt}\right) \cdot \boldsymbol{w}.$$

   Hint: A vector itself is different from a scalar, but the individual components of a vector (in a given coordinate system) are scalars. You can assume that the product rule from calculus applies to regular scalar numbers.

4. An object moves with **constant speed** $v$ along a circle of radius $R$ centered on the origin. At time $t = 0$, the object is at the position $\boldsymbol{r}(t = 0) = (R, 0)$.

(a) What is the period of the object's motion? That is, how long does it take for the object to come back to its original position?

(b) Write down the position vector $\boldsymbol{r}(t)$ of the object as a function of time. *Hint: What is the* ***angle*** *that the position vector makes with the x-axis? How is it related to the* ***arc length*** *of the circular path, and how does the arc length change with time?*

(c) In class, we discussed the two unit vectors, $\hat{x}$ and $\hat{y}$, which were two special vectors with unit length which we chose to describe our coordinate system. However, given any other arbitrary vector $\boldsymbol{r}$, we can define the unit vector $\hat{r}$ to be the vector which points in the same direction as $\boldsymbol{r}$, but with a length of one. To find $\hat{r}$, we simply divide $\boldsymbol{r}$ by its length,

$$\hat{r} = \frac{\boldsymbol{r}}{|\boldsymbol{r}|}.$$

In the case of the position vector of the particle, find the expression for $\hat{r}$ in terms of components.

(d) By differentiating $\boldsymbol{r}(t)$ with respect to time, show that the acceleration of the object is

$$\boldsymbol{a} = -\frac{v^2}{R}\hat{r}$$

where $\hat{r}$ is the unit vector pointing from the origin to the particle. Thus, the particle experiences a uniform acceleration directed radially inwards: this is the *centripetal acceleration* of an object in uniform circular motion.

5. (**Young & Freedman, problem 3.88**) A projectile is thrown from a point P. It moves in such a way that its distance from P is always increasing. Find the maximum angle above the horizontal with which the projectile could have been thrown.

Hint: What is the mathematical equivalent of the statement "its distance from P is always increasing?"

**Some shorter conceptual problems**

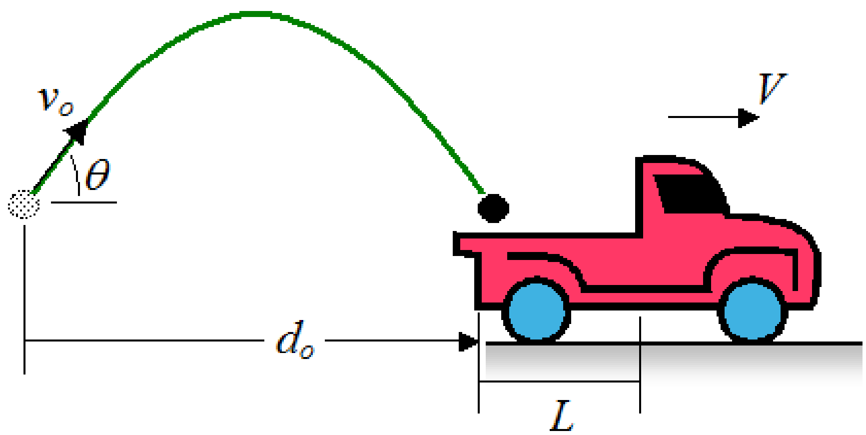
6. In class, I derived some results for the projectile motion of a bullet, in two dimensions. Of course, space is actually three dimensional, so how did I get away with only using two space coordinates?

Hint: Imagine a third coordinate direction perpendicular to the two I used in class. What are the components of $\boldsymbol{r}_0$, $\boldsymbol{v}_0$, and $\boldsymbol{a}$ along this direction?

7. In the same projectile motion problem, I found the time at which the bullet hit the ground, by taking the positive solution in the quadratic formula. What is the physical relevance of the other possible solution for $t$, which is negative?

## Extra Credit Problems

8. A ball is kicked at an angle of $\theta = 45$ degrees, as shown in the figure below. It is intended that the ball lands in the back of a moving truck which has a trunk of length $L = 2.5$ m. If the initial horizontal distance from the back of the truck to the ball, at the instant of the kick, is $d_0 = 5$ m, and the truck moves directly away from the ball at speed v = 9 m/s (as shown), what is the maximum and minimum velocity $v_0$ so that the ball lands in the trunk? Assume that the initial height of the ball is equal to the height of the ball at the instant it begins to enter the trunk.




9. Consider a population of rabbits living on an island. The number of rabbits is a function of time, since as time passes, the rabbits will reproduce and die off at some rate. We'll label the population of rabbits, as a function of time, as $R(t)$.

   (a) An ecologist proposes that the population of rabbits should obey the differential equation
   $$\frac{dR}{dt} = kR,$$
   where $k$ is a constant the does not change with time. Why do you think the ecologist might propose this as a reasonable equation that describes the rate of change of the rabbit population? In terms of the behaviour of the rabbit population, what reproductive facts or hypotheses do you think might be described by such a model? What do you think the meaning of the constant $k$ is, and do you think it should be positive or negative, or could it be both?

   (b) Solve the differential equation using the method of separation mentioned in class. There should be an unknown constant, C, that results from doing an integral at some point.

   (c) Assume that the rabbit population does indeed obey the differential equation. Assume that you also know that at time $t = 0$, the rabbit population is equal to some particular value, $R(t=0) = R_0$. How does this information help you find the value of the unknown constant C? What does the solution of the differential equation look like, written in terms of $R_0$ (instead of C)?

(d) Do you think there is anything unreasonable about the solution? Is there anything about the solution that disagrees with your intuitive sense about how animal populations should behave?

(e) A second ecologist proposes a different model for the population of rabbits, which looks like

$$\frac{dR}{dt} = k\left(1 - \frac{R}{N}\right)R.$$

We will now assume that the constant $k$ is always positive. The number N is another constant, positive value, which does not change over time. What is the right side of the equation approximately equal to when R is much smaller than N? What is the right side approximately equal to when R is much BIGGER than N? How does the sign (positive or negative) of the right side change when we go from one case to the other? What happens when R = N?

(f) Assume that the number of rabbits at time zero is some quantity $R(t=0) = R_0$, and we want to know what the population is at some later time $T$, $R(t=T)$. Solve the differential equation using the method of separation discussed in the appendix of the lecture notes. This time, try performing a **definite** integral on both sides of the equation. What should the upper and lower **bounds** on the integrals be? Does this strategy allow you to avoid the unknown constant C? (You're not required to do the integrals by hand - you can use a calculator, or Wolfram Alpha, if you like).

(g) Assume that $N = 100$ and $k = 1$. Take your solution from the previous part, and make **two** plots of the function $R(t)$ - one of them for $R_0 = 10$, and the other one for $R_0 = 200$. **Make sure the plots extend up until at least $T = 10$.** What is the **major qualitative difference** between the two different plots? Does this shed any light on the meaning of the parameter N?

(h) **Why do you think the second model is a more reasonable model of population growth, as opposed to the first model?**