Physics 20 Homework 2
SIMS 2016

Due: Saturday, August 20$^{\text{th}}$

1. In class, we ignored air resistance in our discussion of projectile motion. Now, let's derive the relevant equation of motion in the case of slow motion through a fluid. In this case, a moving object will feel a drag force proportional to, and in the opposite direction of, its velocity:

$$\boldsymbol{F}_{\text{drag}} = -b\boldsymbol{v}$$

where b is just some proportionality constant that depends on the size and shape of the object and the fluid the object is falling in. This is called *Stokes' law*. For simplicity, we'll work in one dimension.

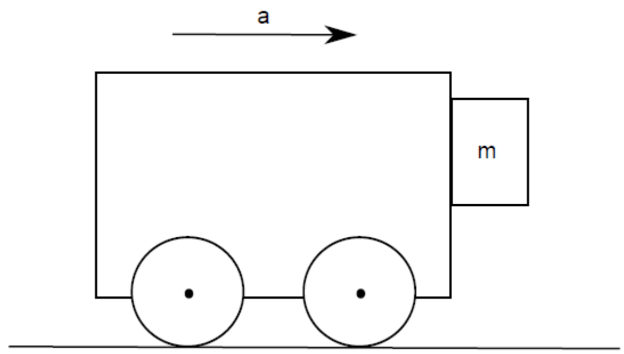
(a) Consider a freely falling particle under the influence of a drag force obeying Stokes' law (it can only move straight up or down, since we're only working in one dimension). Draw a free body diagram for the particle and label all the forces acting on it.

(b) Using your free body diagram, calculate the acceleration of the particle using Newton's second law. What is the critical velocity v_T such that the velocity remains constant? This is called the *terminal velocity* of the particle. What is the terminal velocity in vacuum? (Hint: what is b for an object falling in vacuum?)

(c) In physics class, it's drilled into your head that the time it takes an object to fall to the ground under the influence of gravity is independent of the object's mass. This is true in vacuum, of course, but we all "know" from everyday life that a feather falls to the ground much more slowly than a bowling ball. Using your result from part (b), explain why objects of different masses might take different times to fall to the ground.

(d) **Extra Credit** If you write the acceleration as the time derivative of the velocity, your expression from part (b) becomes a differential equation for the velocity. Solve this differential equation to get the velocity explicitly as a function of time (take the velocity to be v_0 at time $t = 0$). What happens as $t \to \infty$?

2. A block of mass m is placed on the right side of a cart; the cart and the block are accelerating to the right with acceleration a, and the coefficient of static friction between the cart and the block is μ_s. The system is sketched below.



(a) Explain qualitatively why the block won't slip and fall if the acceleration of the cart is large enough.

(b) Calculate the minimum acceleration necessary to keep the block from falling. Start by drawing a free body diagram for the block and labeling all the forces on it; then apply Newton's second law in the x and y directions to calculate the minimum value of a that will prevent the block from falling.

(c) **Extra Credit:** In class, we showed that a block on an inclined plane will not slide as long as the angle of the plane is such that $\tan\theta \le \mu_s$. We can use this result, along with the equivalence principle, to obtain the same result in part (b) a different way.

(i) The equivalence principle states that a uniformly accelerated system is equivalent to a non-accelerating system in the presence of a uniform external gravitational field. In which direction and what magnitude would an external gravitational field need to point to simulate the acceleration $\boldsymbol{a}$? Call this gravitational field $\boldsymbol{g}_{\text{eq}}$.

(ii) Find the net effective gravitational field by adding the gravitational field above to the usual gravitational field $\boldsymbol{g}$:

$$\boldsymbol{g}_{\text{eff}} = \boldsymbol{g}_{\text{eq}} + \boldsymbol{g}$$

What is the angle θ that this field makes with respect to the negative x-axis?

(iii) In the presence of the gravitational field $\boldsymbol{g}_{\text{eff}}$, the block feels as if it were on an inclined plane with angle θ. Using the fact that the block will not slide if $\tan\theta \le \mu_s$, find the minimum acceleration necessary to prevent the block from falling (and make sure it agrees with your result from part (b)!)

3. A boat's **speed** in water is v_{BW}. That is, when the boat travels through the water, its speed *with respect to the water around it* is v_{BW}. This boat starts at point A on one shore of a river and makes its way in a straight line to a point directly across it on the opposite shore (indicated by the dotted line). The **speed** of the river water flowing downstream, with respect to the ground, is v_{WG}. The width of the river is d.

   (a) **In which direction** must the boat be pointing in order to travel to its destination? (Assume $v_{BW} > v_{WG}$). You should specify your answer in terms of the **angle with respect to the dotted line**, as well as the quantities v_{BW} and v_{WG}. **Hint:** How does the boat's velocity with respect to the water relate to its speed in the water, and the angle in question? Use the velocity addition formula to find the velocity of the boat with respect to the ground, in terms of v_{BW} and v_{WG}. What must be true about the velocity with respect to the ground in order for the boat to travel directly across the river?

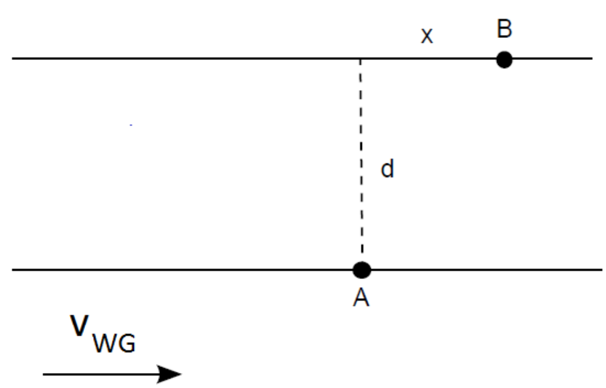
   (b) **How long** will it take the boat to travel directly across the river? Specify your answer in terms of v_{BW}, v_{WG}, and d. Does your answer make sense when $v_{BW} < v_{WG}$? Why or why not? You may want to make use of the trigonometric identity $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$ in order to help simplify your formula.

   (c) Now, imagine instead the boat needs to reach a different point B along the opposite shore, a distance x downstream from where it starts, as shown. **In which direction** must the boat face to get there? Specify your answer in terms of v_{BW}, v_{WG}, x, and d. Make sure your answer reduces to your result in part (a) when $x = 0$! You may find handy the formula

$$A\sin\theta + B\cos\theta = \sqrt{A^2 + B^2}\,\sin(\theta + \delta),$$

   where $\tan\delta = B/A$. **Hint:** Consider the line that extends from point A to point B - what angle does this line make with respect to the dotted line, and how can you relate that angle to x and d? How is this **same** angle related to the components of the boat's velocity with respect to the ground?

   (d) What is the **minimum value** that v_{BW} can have so that the boat is still able to reach the shore at point B?



4. In many physical situations, it is useful to use a **polar coordinate system**, in which the usual basis vectors $\hat{x}$ and $\hat{y}$ are replaced with two other unit vectors $\hat{r}$ and $\hat{\theta}$. These basis vectors are determined in terms of the motion of a particular particle in question, where $\hat{r}$ is the usual unit position vector, and $\hat{\theta}$ is a vector which is perpendicular to $\hat{r}$, oriented counter-clockwise. The figure below illustrates such a coordinate system. Despite the fact that these vectors depend on the location of the particle, and thus change in time, they are still a perfectly good set of basis vectors, and any other vector can be expressed in terms of them.

   (a) What are the expressions for the new basis vectors, in terms of the usual basis vectors? In other words, if I were to write

   $$\hat{r} = f_x(r,\theta)\,\hat{x} + f_y(r,\theta)\,\hat{y}\ ;\ \hat{\theta} = g_x(r,\theta)\,\hat{x} + g_y(r,\theta)\,\hat{y}$$

   what are the functions f_x, f_y, g_x, and g_y?

   (b) What is the expression for the position vector, in terms of these new basis vectors? In other words, if I were to write

   $$\boldsymbol{r} = f_r(r,\theta)\,\hat{r} + f_\theta(r,\theta)\,\hat{\theta},$$

   what are the functions f_r and f_θ?

   (c) What is the expression for the velocity vector, in terms of these new basis vectors? In other words, if I were to write

   $$\boldsymbol{v} = g_r\left(r,\theta,\dot{r},\dot{\theta}\right)\hat{r} + g_\theta\left(r,\theta,\dot{r},\dot{\theta}\right)\hat{\theta},$$

   what are the functions g_r and g_θ? Hint: The simplest way to solve this problem is to work out the components v_x and v_y for the velocity in the usual Cartesian basis vectors $\hat{x}$ and $\hat{y}$ in terms of r, θ, $\dot{r}$, and $\dot{\theta}$, and then compare the resulting vector expression with the two expressions for the new basis vectors $\hat{r}$ and $\hat{\theta}$ from part a.

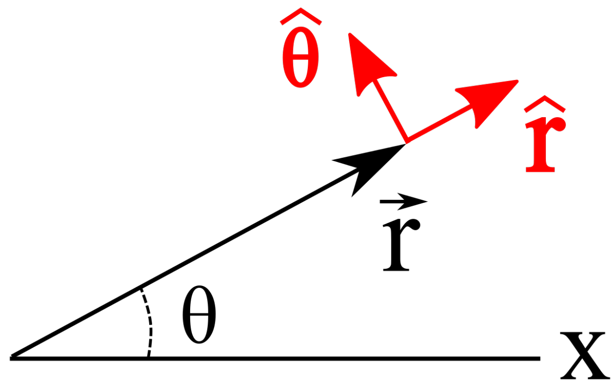
   (d) If we were to naively extend our results from class for the components of the velocity in a Cartesian basis, we might guess that

   $$g_r\left(r,\theta,\dot{r},\dot{\theta}\right) = \frac{d}{dt}f_r(r,\theta),$$

   in which we find the $\hat{r}$ component of the velocity by simply differentiating the $\hat{r}$ component of the position. Is this result correct? Why does this make sense physically? Additionally, we might also naively expect that the same holds true for the $\hat{\theta}$ component,

   $$g_\theta\left(r,\theta,\dot{r},\dot{\theta}\right) = \frac{d}{dt}f_\theta(r,\theta),$$

   in which we find the $\hat{\theta}$ component of the velocity by simply differentiating the $\hat{\theta}$ component of the position. Is this result correct? Why does this make sense physically?



5. Newton's law of universal gravitation states that all matter is attracted to each other, under the influence of a certain force. For two **point** masses of mass m_1 and m_2, separated by a distance r, the magnitude of that force (on each body) is given by

$$F_g = G\frac{m_1 m_2}{r^2},$$

where G is known as Newton's constant, whose value is roughly 6.67×10^{-11} m^3 kg^{-1} s^{-2}. The direction of the force acting on each body is such that the bodies are attracted towards each other. Therefore, the force acting on body one will be directed towards body two, and vice versa. It turns out that the same expression is true for **spherical** bodies, if the distance that we use is the distance between the two centers.

(a) Suppose I take two spherical bodies and hold them at a fixed distance apart. I then let them go, and compute the instantaneous acceleration due to gravity for each body, **at the exact instant I let them go.** Does the acceleration experienced by each body depend on that body's mass? Why or why not?

(b) Suppose that one of the bodies is much more massive than the other, $m_1 \gg m_2$. How do the the gravitational forces on each body compare? What about the accelerations? Again, consider **the exact instant I let them go**.

(c) Because the bodies will begin to move as they experience an acceleration, *the distance between them will start to change, and thus so will the force.* Thus, in general, two massive bodies interacting gravitationally will be subject to changing forces and accelerations as they move throughout space. Using your answer to the previous question, explain why, if one of the bodies is much more massive than the other, we can make the approximation that the much larger body does not move very much, as compared with the lighter body (assuming that we consider sufficiently small enough time periods over which the motion takes place).

(d) If we consider the problem of standing on the Earth and dropping a ball, then it is certainly true that the Earth is much more massive than the ball, and based on our reasoning in the above sections, we can safely assume that over the course of time that the ball is in free fall, the Earth does not move very much. Therefore, we will make the approximation that the Earth stays fixed in space while the ball moves. In terms of the mass of the Earth M, the mass of the ball m, the radius of the Earth R, the height that the ball is initially dropped from h, and Newton's constant G, *write down the expression for the force acting on the ball as soon as I let it go, as well as the resulting acceleration.* Remember that the acceleration has a direction!

(e) Take the expression you found for the acceleration of the ball, and evaluate it when $h = 0$. Using values for the radius of the Earth, the mass of the Earth, and Newton's constant, what is the *numerical* value of this quantity? **Does this value look familiar?**

(f) For any given numbers a and b, whenever b is much less than a, $b \ll a$, it is true that

$$\frac{1}{(a+b)^2} \approx \frac{1}{a^2} - 2\frac{b}{a^3}$$

Use this fact to figure out the *next best approximation* to the acceleration due to gravity near the surface of the Earth. That is, write down an approximate formula for the acceleration due to gravity at the surface of the Earth, which is not just a constant, but which takes into account the height of the ball in a way that is linear in the variable h. Do you think this extra, first order term would be important in most practical situations? Why or why not? (Hint: rearrange your expression for the first order term in a way that emphasizes the relation between h and R).

(g) **Extra Credit:** Explain why the approximation formula I gave you above is true, using a Taylor series expansion.

(h) **Extra Extra Credit:** Solve for the motion of the ball as a function of time, using the slightly more correct form of the acceleration (the one you found above, which is correct up to linear order in h). **You'll probably need to ask me for help on this one.**

Note: From what I could gather from the diagnostic exams, it looks like a decent number of you know some basic information about how to do a Taylor series. If this is not true, but you would like to learn about them, then a very good introduction to the subject can be found on Wikipedia. There is also a short introduction to Taylor Series in the Appendix of your lecture notes. If you would like some help from me on doing a Taylor series expansion, you can feel free to ask me for help during office hours.

6. **Extra Credit** Without using any math, use the gravitational equivalence principle to deduce the conclusion of homework zero. That is, use the equivalence principle to explain why the bullet will always hit the target.