Physics 20 Homework 3
SIMS 2016

Due: Thursday, August 25$^{\text{th}}$

Special thanks to Sebastian Fischetti for problems 1, 5, and 6. Edits in red made by Keith Fratus.

1. The *ballistic pendulum* is a device used to measure the speed of a bullet. A bullet of mass m is fired into a block of mass M hanging on a pendulum of length L, into which it embeds itself; this causes the pendulum to swing up to some maximum angle θ, as shown. Calculate the initial speed of the bullet in terms of m, M, θ, L, and g. Hint: What conservation laws might you be able to apply in this problem? When can you apply each one, and when can you not?

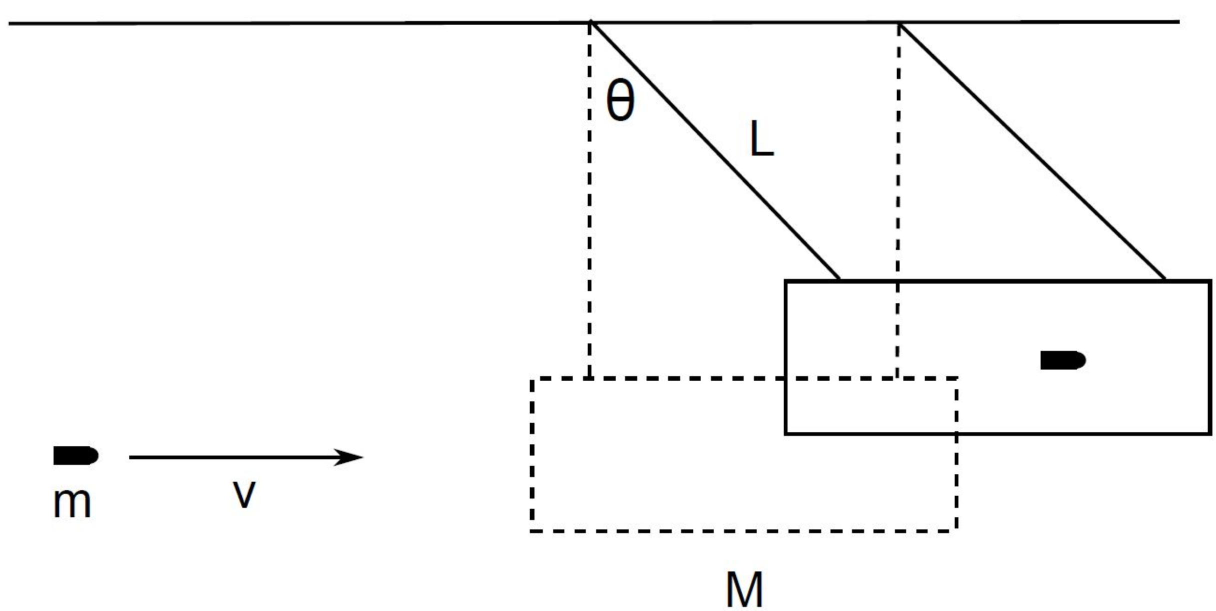


Figure 1: The operation of a ballistic pendulum. Special thanks to Sebastian Fischetti for the figure.

2. In lecture, we discussed the definition of potential energy for a single particle which is subjected to an external force. However, we know that in reality, forces arise from the interactions between multiple particles. It is possible to generalize the definition of potential energy to describe two particles interacting with each other as a closed system. As an example, the potential energy of a pair of neutral atoms can be modelled, to a very good approximation, by the *Lennard-Jones potential*, given by

$$U(r) = U_0 \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right]$$

Here U_0 and r_0 are constants, and r is the distance between the two atoms. Despite the fact that the atoms are electrically neutral overall, the charge due to the protons and electrons is spread out over a non-zero region, so there is still a small, residual electrical interaction between the two. In this special case in which the potential energy only depends on the distance between the two particles, the *force* acting on a given atom is given by

$$\boldsymbol{F} = -\frac{dU}{dr}\,\hat{r},$$

where $\hat{r}$ is the vector pointing from that body, towards the other one.

(a) Using this potential energy function, find the force and equilibrium distance between the two atoms.

(b) Sketch a graph of the Lennard-Jones potential, labeling any relative extrema (your results from part (a) may help).

(c) The atoms are pushed closer together until the distance between them is R. Explain why there exists a critical inter-atomic distance σ such that when the atoms are released from rest, if $R > \sigma$, the atoms will oscillate back and forth, while if $R < \sigma$, the atoms will fly apart to infinity. Find the value of σ in terms of r_0 (Hint: what is the total energy of the system? Hint # 2: look at your graph from part (b)!).

(d) By examining the overall shape of the above potential energy function, explain why physical objects tend to expand in size when they are heated. Hint: What does heating an object do to its internal energy at a microscopic level?

3. (**Young & Freedman, problem 6.102**) An airplane in flight is subject to an air resistance force proportional to the square of its speed,

$$F_{\text{drag}} = \alpha v^2$$

where α is some constant. This force points opposite to the velocity. If we assume that the plane flies on a level path, this velocity points entirely in the horizontal direction, since the plane is not moving up or down. But there is an additional resistive force because the airplane has wings. Air flowing over the wings is pushed down and slightly forward, so from Newton's third law the air exerts a force on the wings and airplane that is up and slightly backward, as shown below. The upward force is the lift force that keeps the airplane aloft, and the backward force is called *induced drag*. At flying speeds, the induced drag is inversely proportional to v^2:

$$F_{\text{induced}} = \beta / v^2$$

where β is another constant. Thus the **total air resistance force** can be expressed by

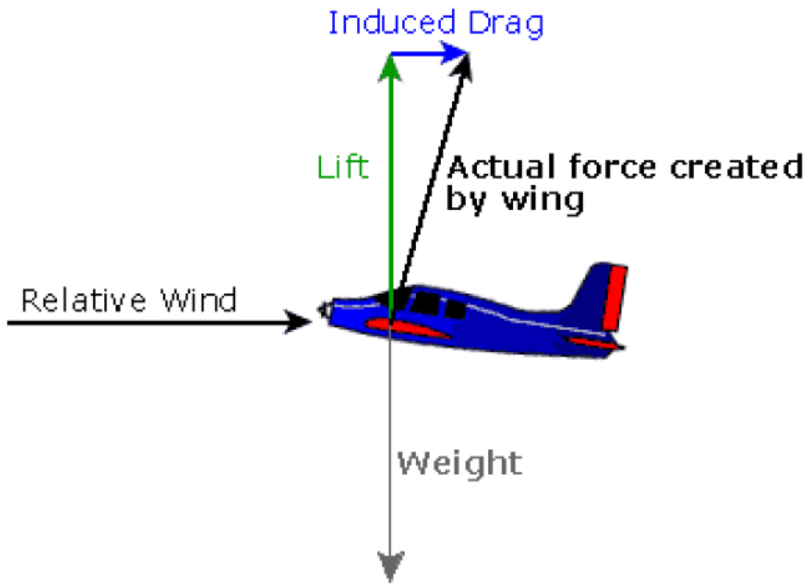
$$F_{\text{air}} = \alpha v^2 + \beta / v^2$$

In steady flight, the engine must provide a forward force that exactly balances the air resistive force.

(a) Calculate the speed at which the airplane will have the maximum ***range*** (that is, will travel the greatest distance) for a given quantity of fuel.

(b) Calculate the speed at which the airplane will have the maximum ***endurance*** (that is, will remain in the air the longest time) for a given quantity of fuel.

Hint: The amount of fuel that the engine uses during some amount of time is proportional to how much work the engine does in that amount of time. The amount of fuel the plane can carry is some finite amount, so the total amount of work the engine can perform is some finite amount. The rate at which the engine performs this work is the power. How does the power supplied by the engine relate to the force it supplies to the plane, and the velocity at which the plane travels? If the engine supplies some amount of power for some amount of time, how much work is done during that time? You'll need to read the last page of my notes on work and kinetic energy to find this information about power. Your final answers for parts a and b won't actually depend on the amount of fuel the plane is carrying with it.


Induced Drag
Lift
Actual force created by wing
Relative Wind
Weight

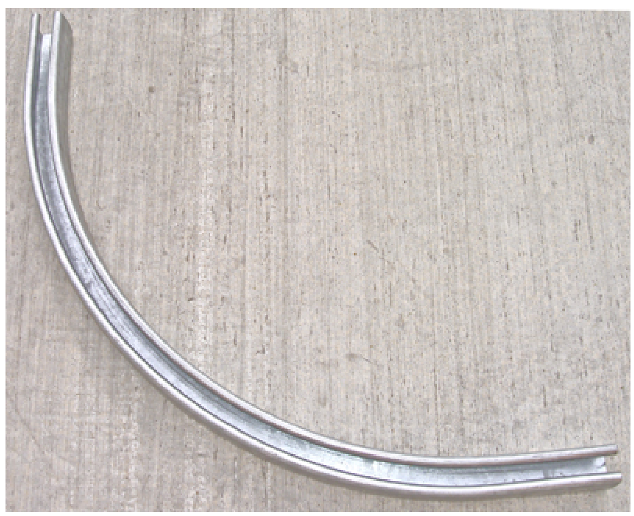
4. Imagine a situation similar to the one described in lecture, in which an object is given an initial shove, and is allowed to move along the floor until it comes to rest. However, this time, the object is constrained to move along a particular path, due to moving along some sort of guiding track. This might be something similar to the guiding track in the image below. In particular, we assume that if we set up a coordinate system at the initial position of the block before it receives a shove, the *shape* of the guiding track it is constrained to move along is given by the function

$$y = x^2.$$

This is illustrated below in the figure. Additionally, the floor on which the block is moving has been sanded down in a very strange way. The coefficient of kinetic friction between the block and the floor varies depending upon where we are, in such a way that

$$\mu_k = A\sqrt{x^2 + 4y^2},$$

where A is some constant number with units of inverse length.



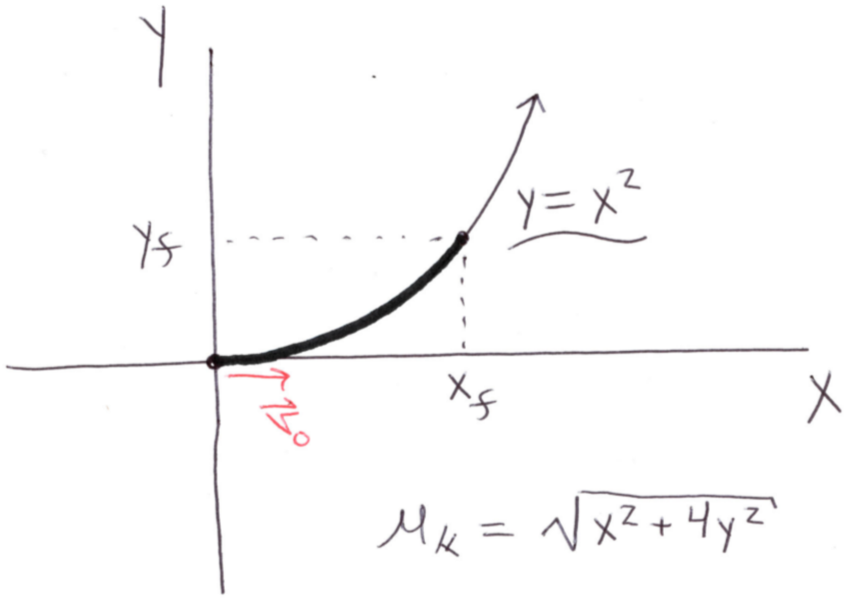
After we give the block some initial shove with an initial speed v_0, it travels for some distance along the guiding track, before coming to rest at some final position

$$\boldsymbol{r}_f = (x_f, y_f).$$

We would like to compute how much work was done as it travelled along the track, using our totally general formula for work,

$$W = \int \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{v}(t)\ dt.$$

Unfortunately, as the block was travelling, we didn't pay attention to what its trajectory was, so we're not sure what $\boldsymbol{r}(t)$ was. However, there is still a way we can compute the work, without knowing exactly what the position as a function of time was. Let's see in detail how we can do this.



(a) Let's assume that the block starts moving at time $t = 0$, comes to rest at some later time $t = T$, and that its x coordinate evolves according to

$$x(t) = \left(\frac{t}{T}\right) x_f.$$

That is, let's assume the x coordinate evolves linearly from 0 to x_f. *This is probably not the way the x coordinate actually evolves in time, but let's assume it is anyway.* In this case, what must $y(t)$ be equal to, so that the block moves along the track?

(b) Assuming the block follows the trajectory in part a, what is the velocity, $\boldsymbol{v}(t)$? Remember that velocity is a vector!

(c) Remember that as the block moves, the frictional force always points in the opposite direction of the velocity, so that the force acting on the block is

$$\boldsymbol{F} = \boldsymbol{f_k} = -f_k \hat{v},$$

where $\hat{v}$ is a unit vector that points in the direction of the velocity. Compute what this force vector is equal to, in terms of t, T, x_f, A, m (the mass of the block), and g (the acceleration due to gravity). Remember that f_k depends on μ_k, which depends on position, and that the position depends on time.

(d) Now that you have explicit expressions for $\boldsymbol{F}$ and $\boldsymbol{v}$, as functions of time, you can compute the work integral

$$W = \int_0^T \boldsymbol{F}(t) \cdot \boldsymbol{v}(t) \; dt.$$

Perform this integral, and express your answer in terms of the relevant physical quantities. Does your answer depend on T?

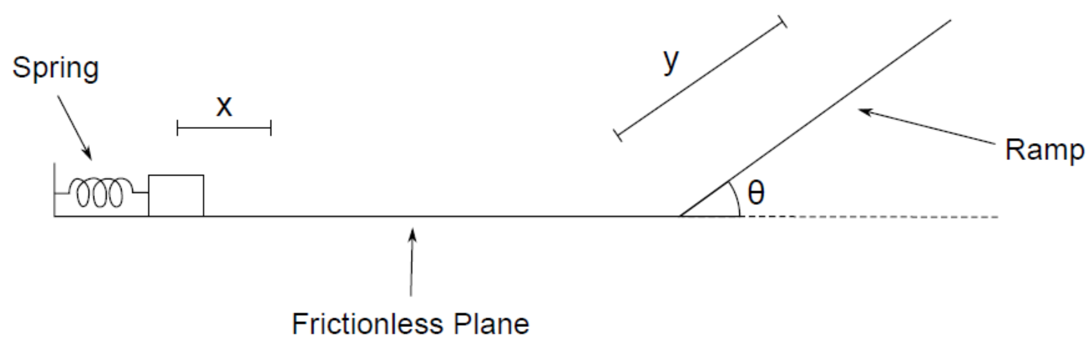
(e) Repeat the previous steps, except this time, assume that the x coordinate evolves according to a *different* function,

$$x(t) = \left(\frac{t}{T}\right)^2 x_f.$$

The time coordinate still evolves from $t = 0$ to $t = T$, and the x coordinate still arrives at $x = x_f$ when $t = T$. What answer do you find for the total work? *Is it the same as the value you found when you made a different assumption about the block's trajectory?* The result you've found, that different possible trajectories give the same answer for the work, is called, in fancy technical terms, *reparametrization invariance*. It is a very general result, and is always true, so long as $\boldsymbol{F}$ only depends on the position of the block. Proving this, however, is a bit beyond the scope of this course.

(f) What must the initial speed, v_0, be equal to, in order for the block to arrive at some final x coordinate x_f? In other words, solve for the initial speed v_0 in terms of x_f and the other relevant quantities in the problem.

5. A block lies on a flat, frictionless plane against a spring with spring constant k compressed from its equilibrium position by a distance x. At the other end of the plane is a ramp inclined at an angle θ. The coefficients of friction (both kinetic and static) of the ramp vary with the distance along it as $\mu(y) = Ay$, where A is some constant and y is the distance from the bottom of the ramp; that is, the ramp is frictionless at the bottom, and the coefficients of friction increase linearly as you move up the ramp (here, we're assuming the coefficients of friction are equal: $\mu_k = \mu_s = \mu$). The system is sketched below.

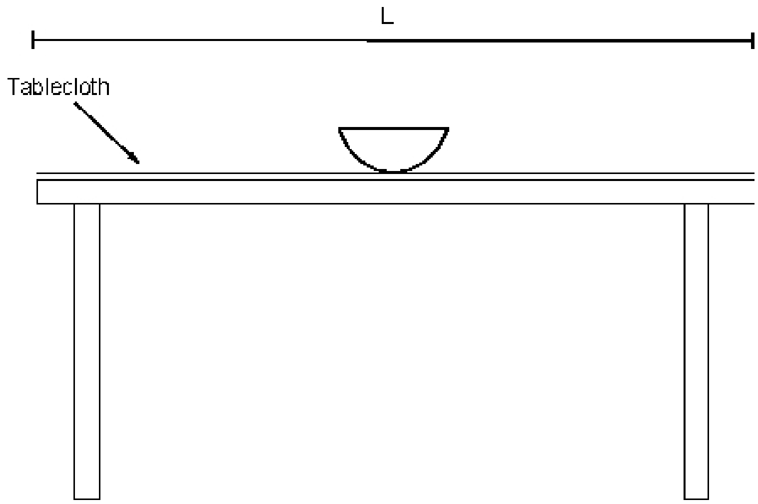


The block is released and pushed along the frictionless plane by the spring, and then moves up the ramp. By using energy methods (i.e. conservation of energy, the work-energy theorem), show that when the block comes to rest on the incline, it will remain at rest if

$$x^2 \geq \frac{3mg\sin^2\theta}{Ak\cos\theta}$$

Hint: what determines whether a block at rest on an incline will begin to slide down or not?

6. **Extra Credit** A bowl of mass m rests on a table cloth in the middle of a table of length L, as shown. The table cloth is then pulled out at speed v from underneath the bowl. Assume that the table cloth slides under the bowl; the coefficient of kinetic friction between the bowl and the cloth is μ_1, and the coefficient of kinetic friction between the bowl and the surface of the table is μ_2.



   (a) Discuss qualitatively the behavior of the bowl, and explain why if the table cloth is pulled fast enough, the bowl will not fall off the table (Hint: there are two "stages" in this problem: when the table cloth is being pulled out from underneath the bowl, and when the bowl is sliding on the table. What is the motion of the bowl during each stage?)

   (b) Let v_c be the critical speed necessary for the table cloth to slide out from under the bowl and for the bowl to come to rest right at the edge of the table. Find an equation relating v_c to the given quantities in the problem (that is, find an equation that contains v_c, μ_1, μ_2, L, and g, without finding v_c explicitly) (Hint: what is the net work done on the bowl during this process? What is/are the only force(s) doing work on the bowl?). **Extra Extra Credit:** Calculate v_c.

7. **Extra Credit** Imagine a rocket which is out in empty space, so that there is negligible gravity. The initial mass of the rocket is m_0. After starting from rest, the rocket begins burning fuel in order to accelerate at a **constant** rate a, until it reaches a final speed of v_f. The rocket has some mass $m(t)$, and some velocity $v(t)$. This is illustrated in the figure below. We are interested in finding the **total work done by the rocket's engines.**

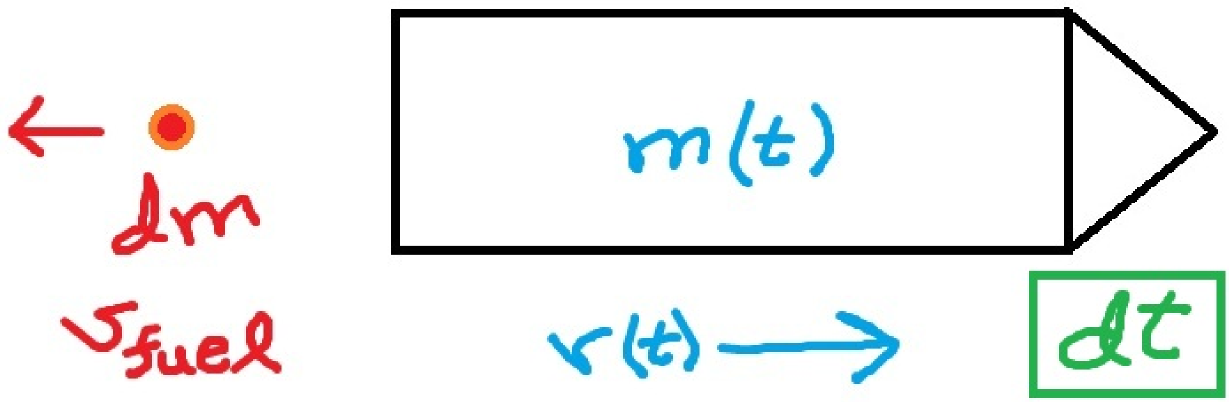


Figure 2: The motion of the rocket as it releases a small amount of fuel in a time dt.

(a) Show that the work done on the **rocket itself** is given by

$$W_r = \int_0^{v_f/a} \left(ma + at\frac{dm}{dt} \right) at\, dt.$$

Hint: Start with the basic definition of work

$$W_r = \int_0^d F(x)\, dx = \int_0^{t(d)} F(t)\, v dt$$

and remember Newton's second law

$$F(t) = \frac{dp}{dt} = \frac{d}{dt}(mv).$$

(b) Show that the total work done on the fuel exhausted out the back is

$$W_{fuel} = -\int_0^{v_f/a} \frac{dm}{dt} \left(v(t) - v_{\text{ex}} \right)^2 dt.$$

Hint: For an infinitesimal piece of exhaust being ejected out the back,

$$dW_{fuel} = F(t)\, v_{fuel}(t)\, dt = \frac{dp_{fuel}}{dt} \left(v(t) - v_{\text{ex}} \right) dt,$$

where $v(t)$ is the velocity of the rocket, and v_{ex} is the speed that the exhaust is emitted from the rocket, with respect to the rocket. Remember that the mass of the infinitesimal amount of ejected fuel is dm.

(c) Add these two expressions to find

$$W_T = \int_0^{v_f/a} ma^2 t + \frac{dm}{dt}\left(2atv_{\text{ex}} - (v_{\text{ex}})^2\right)\ dt$$

(d) Take the rocket equation we found in class

$$v - v_0 = v_{\text{ex}} \ln\left(\frac{m_0}{m}\right),$$

and differentiate both sides with respect to time, in order to find

$$\frac{dv}{dt} = -\frac{v_{\text{ex}}}{m}\frac{dm}{dt}.$$

Use this differential equation to find $m(t)$. **Hint:** Use the chain rule, and remember the acceleration is constant.

(e) Use this expression for $m(t)$ to explicitly compute the work integral, and show that

$$W_T = m_f v_{\text{ex}} v_f.$$

(f) **Extra Extra Credit:** There is a common mistake people tend to make when computing the total work done by the rocket's engines in the previous problem. They decide to find the kinetic energy of each piece of fuel that leaves the rocket, integrate this to get a total, and then add it to the final kinetic energy of the rocket. But this actually does not give the correct answer, and the reason is because you are missing some work which is done when thinking about it that way. Explain exactly where this missing energy went.

8. **Challenge Problem** The subject of scattering is incredibly important in physics. In a typical scattering experiment, a source of highly energetic particles is aimed at a target material, and by measuring the scattering angles of different particles, we can learn something about the target material. A rough sketch of such an experiment is shown in Figure 3. In most high energy particle experiments, this is the way that we actually deduce the structure of subatomic particles like atoms - we scatter lighter particles off of them with some impact parameter, and measure their deflection. This is how Rutherford figured out the structure of the atom in his famous gold foil experiment.

   In this problem we're going to study this subject in a quantitative way for a simple two-dimensional system. Consider the physical setup in Figure 4. The red dot is a particle sitting still at the origin of some coordinate system we've set up, and the purple dot is an incoming particle. The incoming particle comes in from very far away, and when it is very far away, it is traveling in a straight line with a constant velocity $\boldsymbol{v}_0$. When it is very far away, its y coordinate above the origin is equal to b, a number which is called the ***impact parameter***.

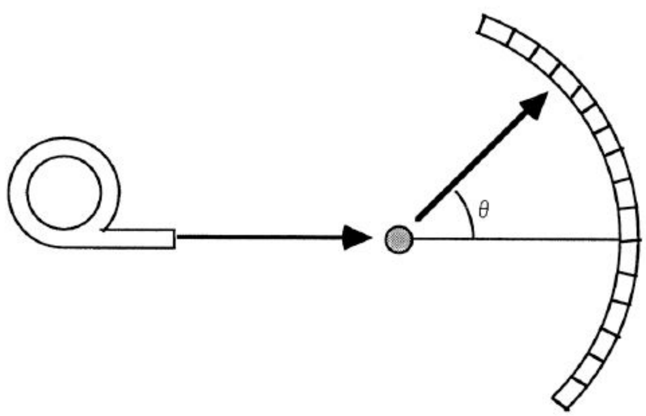


Figure 3: A typical scattering experiment between two particles. Figure taken from "Quantum Mechanics: A Modern Development," by Leslie Ballentine.

   We assume that the red particle is much, much heavier than the incoming one, and so it effectively sits still at the origin. The purple dot interacts with the red dot, and as a result, is subjected to a potential $U(r)$, which only depends on the distance from the origin. The interaction between the particles is conservative, so that the total energy of the incoming particle,

$$E = \frac{1}{2}mv^2 + U\left(r\right) \tag{1}$$

   is conserved. We assume that the potential is positive, so that the two objects repel each other. The blue line shows the path that the incoming particle would take if there were no interaction, while the dashed purple line shows the trajectory that the incoming particle will take as it is scattered away.

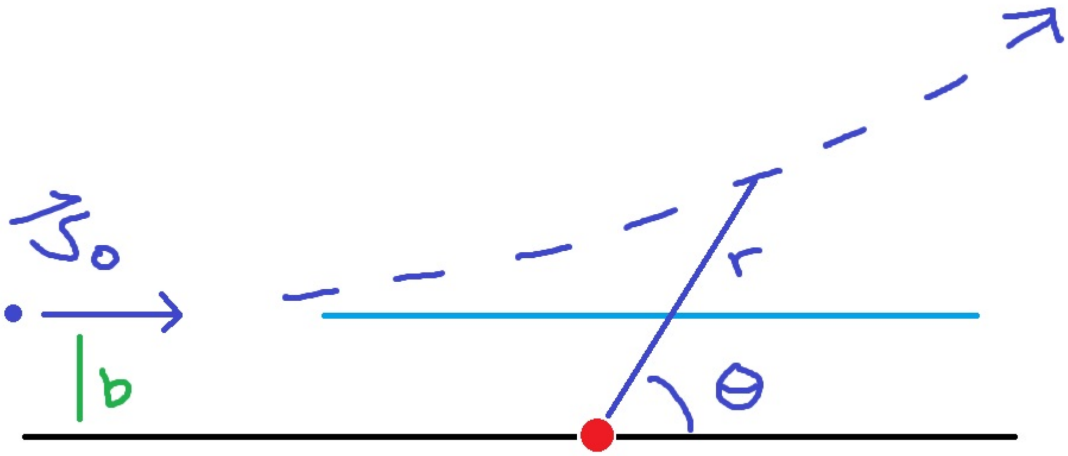


Figure 4: A typical scattering experiment between two particles.

(a) In this problem, it will be most useful to describe our system using the angle θ, along with the distance between the two particles, r. Derive an expression for the kinetic energy in terms of r, θ, and their time derivatives. Use this expression to write the total energy in terms of r, θ, and their time derivatives. *Hint: The easiest way to do this is to first set up a coordinate system with the usual x and y coordinates, write down the position vector in terms of r and θ, and then use the chain rule to find the velocity vector in terms of r, θ, and their time derivatives.*

(b) We will now define a quantity known as the ***angular momentum***,

$$J = mr^2\dot{\theta} = mr^2\frac{d\theta}{dt}. \tag{2}$$

It turns out that whenever we have a potential energy that **only depends on the quantity** r, and has no dependence on θ, then the quantity J will always be conserved. That is to say, it does not change over time. In this problem, what is the angular momentum of the incoming particle? **Express your answer in terms of** m**,** b**, and** v_0. *Hint: To find the angular momentum, you can make use of the fact that so long as the potential energy only depends on r, the angular momentum is always conserved no matter what. If we have a particle that is initially coming in from very far away, then it will always keep the same angular momentum it has from the beginning, whether or not the scattering red dot is there. Thus, the angular momentum stays the same over the course of its motion, whether it follows* ***the blue line or the purple dotted line****. Calculating the angular momentum* *is easiest by pretending the particle follows the blue line instead, and considering the point in time when the y coordinate is equal to the radius, and the x coordinate is zero. That is, try calculating the angular momentum when the purple particle is sitting above the red particle. In order to do this, it's useful to find the angle in terms of b, v_0, and t, and take a time derivative.*

(c) Use the expression you found for the angular momentum to eliminate $d\theta/dt = \dot{\theta}$ from the expression for the total energy. Write the expression for the total energy in terms of the radius r, along with m, b, and v_0, the initial speed.

(d) You should now have the total energy in a form which looks like

$$E = \frac{1}{2}m\dot{r}^2 + U(r) + f(r), \quad (3)$$

where the last piece, $f(r)$, depends on the radius, as well as the angular momentum (which you expressed in terms of the other parameters of the problem). Notice that if we define a new function,

$$U_{\text{eff}} = U(r) + f(r), \quad (4)$$

then mathematically, our problem looks like a particle moving in just one dimension, described by r, under the influence of an *effective potential*. The *rotational symmetry* of the problem, or the fact that the potential energy only depends on the radius r, has allowed us to effectively reduce the number of dimensions we need to focus on. **What must be true about the value of the effective potential when the incoming blue particle is at its point of closest approach to the red particle** (that is, when the distance between the two particles is a minimum)?

(e) Using the chain rule,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}, \quad (5)$$

write the total energy **only** in terms of r, constant parameters, and the derivative $dr/d\theta$.

(f) If you rearrange the expression you found above, you should be able to write something that looks like

$$\frac{dr}{d\theta} = \pm h(r), \quad (6)$$

where $h(r)$ is some function of r, and also depends on various constant parameters of the problem. The $\pm$ sign should come from having taken a square root. Now consider breaking the motion of the particle up into two stages - the first stage is when the purple particle comes in towards the red particle, and the second stage is when it scatters away after having reached its point of closest approach. Which stage of the motion should correspond to the plus sign, and which stage should correspond to the minus sign? *Hint: During each stage, is the radius decreasing, or increasing? Likewise, is the angle decreasing, or increasing? What should the sign of the derivative be if both are increasing or decreasing together? What should it be if one is increasing while the other is decreasing?*

(g) The expression in the previous part can be viewed as a differential equation relating the radius and angle. Using the usual techniques of separation, **derive an expression for the angle of the particle when it is at its point of closest approach, and has reached the end of the first stage of motion**. Specify your answer in terms of the quantity r_*, the radius of closest approach. Your answer should be in the form of an integral over some function of radius, with boundary terms that involve the quantity r_*. *Hint: This is fundamentally no different from the previous differential equations you have solved - during the first stage of the motion, the angle increases as a function of radius, and you want to know the value of the angle at some particular radius r_*.* ***What is the initial angle?***

(h) Again, use the method of separation to study the motion of the particle during the second phase of its motion. Separate the differential equation (remembering to use the opposite sign now!) to compute the total change in angle as the particle undergoes the second stage of motion. **Add this value to the result from the previous section to find the final outgoing angle after the particle has scattered away from the red dot.** Your answer will still be specified in terms of the radius of closest approach, r_*.

(i) For the case that the potential is the Coulomb potential,

$$U(r) = \frac{\alpha}{r}, \tag{7}$$

where α is some parameter describing the interaction of the particles. For this choice of potential, what is the radius of closest approach?

(j) For the case of the Coulomb potential, find the final scattering angle by explicitly performing the integral expression in the previous sections. You may use Mathematica or a calculator to do this.

You have now derived the famous Rutherford scattering formula, one of the most important results in all of physics.