Physics 20 Homework 1
SIMS 2016

Due: Wednesday, August 17$^{\text{th}}$

Problem 1

The idea of this problem is to get some practice in approaching a situation where you might not initially know how to proceed, and need to get a sense of how to assess what is relevant in the problem.

Human nutrition is of course a complicated (and sometimes controversial) subject. However, we can probably assume that as a basic approximation, the driving factor influencing someone's required diet is that they need to consume a certain number of calories each day. This will vary to some extent based on an individual's diet, but the average number quoted on nutritional food labels is 2,000 calories. While this number may fluctuate somewhat, we know it's certainly not 200, and definitely not 20,000, so we should be able to safely make some order of magnitude estimates with this number.

Now, it turns out that an ear of corn contains about 60 calories, which is a number documented in a few different sources. If we round this number and say that 50 calories per ear of corn is close enough, then in order to consume 2,000 calories of corn each day, a human would need to eat 40 ears of corn each day. Multiplying this by an approximate world population of 7 billion people, we find that each day, about 280 billion ears of corn would be consumed.

If we assume that the world's consumption of food is roughly distributed evenly over the day, then dividing this by the number of seconds in a day (86,400), we find that each second, the human population would be consuming about 3.2 million ears of corn.

Now, certainly this end result would depend on what specific assumptions we made regarding the above numbers, but it should certainly be roughly correct. The consumption of corn would definitely not be one thousand ears per second, nor would it be one billion - these numbers are much too low and high, respectively. While this example may seem silly, this type of calculation is very important in physics all the time, and plenty of other fields as well. Often times this sort of reasoning can get us an answer very quickly, without wasting time doing further calculations. For example, imagine some enterprising company had developed a scheme to profit off of the world's consumption of corn, but for some reason they had decided that it would only be feasible if the world consumption of corn was one billion ears per second. Well, we can see that even if the world ate nothing but corn, that number is still a thousand times more than the world consumption. Looks like the company should come up with another plan...

Problem 2

If $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$, then we have

$$\begin{aligned} c^2 &= \boldsymbol{c} \cdot \boldsymbol{c} \\ &= (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) \\ &= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b} + 2\boldsymbol{a} \cdot \boldsymbol{b} \end{aligned}$$

But $\boldsymbol{a} \cdot \boldsymbol{a} = a^2$ and likewise for $\boldsymbol{b}$, and by definition of the dot product, $\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta$, so

$$\boxed{c^2 = a^2 + b^2 + 2ab\cos\theta}$$

as desired. Now, note that $\cos\theta$ attains a maximum value of 1 (when $\theta = 0$, corresponding to $\boldsymbol{a}$ and $\boldsymbol{b}$ being parallel), so we have that

$$2ab\cos\theta \leq 2ab$$

and therefore,

$$c^2 = a^2 + b^2 + 2ab\cos\theta \leq a^2 + b^2 + 2ab = (a+b)^2$$

Since a, b, and c are all positive (they're magnitudes of vectors), we can take the positive square root of both sides to obtain

$$c \leq a + b$$

or

$$\boxed{|\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|}$$

as desired (here, I simply wrote by definition $a = |\boldsymbol{a}|$, $c = |\boldsymbol{c}| = |\boldsymbol{a} + \boldsymbol{b}|$, etc.)

Problem 3

If we write the expression for the dot product in terms of components, we have

$$\frac{d}{dt}\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)=\frac{d}{dt}\left(v_x w_x+v_y w_y+v_z w_z\right),$$

or

$$\frac{d}{dt}\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)=\frac{d}{dt}\left(v_x w_x\right)+\frac{d}{dt}\left(v_y w_y\right)+\frac{d}{dt}\left(v_z w_z\right).$$

Now, since the individual components are simply numbers, the product rule from calculus applies to them. In other words,

$$\frac{d}{dt}\left(v_x w_x\right)=\frac{dv_x}{dt}w_x+v_x\frac{dw_x}{dt},$$

and likewise for the other components. Therefore, we can write

$$\frac{d}{dt}\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)=\frac{dv_x}{dt}w_x+v_x\frac{dw_x}{dt}+\frac{dv_y}{dt}w_y+v_y\frac{dw_y}{dt}+\frac{dv_z}{dt}w_z+v_z\frac{dw_z}{dt}.$$

Now, if we regroup terms slightly, we can write

$$\frac{d}{dt}\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)=\left(v_x\frac{dw_x}{dt}+v_y\frac{dw_y}{dt}+v_z\frac{dw_z}{dt}\right)+\left(\frac{dv_x}{dt}w_x+\frac{dv_y}{dt}w_y+\frac{dv_z}{dt}w_z\right).$$

If we recall that

$$\frac{d\boldsymbol{v}}{dt}=\left(\frac{dv_x}{dt},\frac{dv_y}{dt},\frac{dv_z}{dt}\right),$$

and likewise for $\boldsymbol{w}$, then it becomes clear that our expression is equal to

$$\frac{d}{dt}\left(\boldsymbol{v}\cdot\boldsymbol{w}\right)=\boldsymbol{v}\cdot\left(\frac{d\boldsymbol{w}}{dt}\right)+\left(\frac{d\boldsymbol{v}}{dt}\right)\cdot\boldsymbol{w}.$$

Problem 4

(a) The circumference of the circle is $2\pi R$. If the object is moving at speed v, then the time it takes for the object to go once around the circumference of the circle is

$$\boxed{T = \frac{2\pi R}{v}}$$

(b) The object is traveling in a circle of radius R centered on the origin, so we should be able to parametrize its motion as

$$x(t) = R\cos\theta(t)$$
$$y(t) = R\sin\theta(t)$$

where $\theta(t)$ is the angle the object makes with the x-axis at time t. Now, since the object starts at the position $\boldsymbol{r}(0) = (R, 0)$, we have that $\theta(0) = 0$. Since the object must return to its initial position after a time T, we must also have that $\theta(T) = 2\pi$. Since θ must increase linearly in time (can you explain why this is true by thinking about how arc length is related to angle?), we can therefore conclude that $\theta(t) = 2\pi t/T = vt/R$, and thus

$$\boxed{\boldsymbol{r}(t) = (x(t), y(t)) = R(\cos(vt/R), \sin(vt/R))}$$

(c) Differentiating once, we get that the velocity of the object is

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt} = v(-\sin(vt/R), \cos(vt/R))$$

Differentiating once more, we get that the acceleration is

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} = -\frac{v^2}{R}(\cos(vt/R), \sin(vt/R))$$

But note that we can write the unit radial vector $\hat{r}$ as

$$(\cos(vt/R), \sin(vt/R)) = \frac{\boldsymbol{r}(t)}{R} \equiv \hat{r}$$

and therefore

$$\boxed{\boldsymbol{a} = -\frac{v^2}{R}\hat{r}}$$

as desired.

Problem 5

Let's assume that we've placed the origin of a coordinate system at the point P. Using our results from lecture, this corresponds to a projectile motion problem with $h = 0$, so that the position of the object as a function of time is given by

$$\boldsymbol{r}(t) = \begin{pmatrix} v_0 \cos\theta t \\ v_0 \sin\theta t - \frac{1}{2} g t^2 \end{pmatrix}$$

Now, the mathematical equivalent of the statement "its distance from P is always increasing" is that *the derivative of the distance from P is always greater than zero.* In more mathematical terms, we require

$$\frac{dD}{dt} = \frac{d}{dt}\left(\sqrt{r_x^2 + r_y^2}\right) > 0.$$

Now, we could compute the derivative of D, but if we think for a moment, we realize that because D is a strictly positive quantity, if D is always increasing, that must mean that D^2 is *also* always increasing. It's a little bit easier to take a derivative of D^2 than it is to take a derivative of D, so we'll revise our statement a little bit to be

$$\frac{dD^2}{dt} = \frac{d}{dt}\left(r_x^2 + r_y^2\right) > 0.$$

Using our results for projectile motion, this statement becomes

$$\frac{dD^2}{dt} = \frac{d}{dt}\left(v_0^2 \cos^2\theta t^2 + v_0^2 \sin^2\theta t^2 - g v_0 \sin\theta t^3 + \frac{1}{4} g^2 t^4\right) > 0,$$

or,

$$\frac{dD^2}{dt} = \frac{d}{dt}\left(v_0^2 t^2 - g v_0 \sin\theta t^3 + \frac{1}{4} g^2 t^4\right) > 0.$$

Taking the derivative, we have

$$2v_0^2 t - 3 g v_0 \sin\theta t^2 + g^2 t^3 > 0.$$

Now, since we are always interested in times in the future, t is always positive, and so we can divide both sides of the inequality by t, in order to find

$$2v_0^2 - 3 g v_0 \sin\theta t + g^2 t^2 > 0.$$

We now have an inequality which is quadratic in time. What we want to be the case is that *there is never a time at which this quantity is negative*, which is to say, there is never a time at which the distance is decreasing. In order for this to be true, *there must never be a root which solves the quadratic equation*

$$2v_0^2 - 3 g v_0 \sin\theta t + g^2 t^2 = 0.$$

When does a quadratic equation never have a solution? If the solutions to the quadratic equation are given by

$$t_{\pm} = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right),$$

then we know that a real solution to the quadratic equation never exists when the term in the square root of the quadratic formula is negative,

$$b^2 - 4ac < 0.$$

In our case, this condition becomes

$$9g^2 v_0^2 \sin^2\theta - 8g^2 v_0^2 < 0,$$

or

$$\sin^2\theta < \frac{8}{9}.$$

The threshold angle thus becomes

$$\theta = \arcsin\left(\sqrt{\frac{8}{9}}\right) \approx 70.53^\circ$$

Intuitively, we know that if we fire a projectile horizontally, it will always move away from us. We also know however that if we fire a projectile straight up, its distance to us will certainly decrease at some point as it falls back down towards us. Our answer tells us that the threshold angle in between these two limiting extremes is about 70.53 degrees. Notice that this number is universal! It does not depend on the acceleration due to gravity, or the initial speed of the projectile.

Problem 6

If I were to set up this hypothetical third direction, I have some freedom as to how to shift its axis. For the sake of simplicity, if I call this the z direction, then I'll say that $z = 0$ is the location of the bullet initially. However, I know that I've chosen my initial velocity to point along the x and y directions, which is a choice I can always make, since I'm choosing how to orient my axes, and I do so such that they point along these two directions alone. So there is no z component of the initial velocity either.

Now, the important question is what the component of acceleration is along this third direction. While I can choose to orient my axes so that the initial position and velocity have no z component, if there is an acceleration along the z direction, then they will not stay this way. However, my third direction, if it is perpendicular to x and y, would point along the surface of the Earth. Physically, I know that gravity only exerts a force downwards, and so there is also no acceleration along this third direction.

Because there is no initial velocity or acceleration along this third direction, then the component of the position along the direction always stays zero. So, while it is technically still there, it always stays the same, and so it is not really interesting to us. Notice that this would not have been the case if there were some other force (for example maybe an Electric force) pointing along the surface of the Earth, in a direction *different* from the initial velocity of the bullet. Then I would need all three directions to specify the trajectory of the bullet.

What we say in this case is that the surface of the Earth has *rotational symmetry*, in the sense that as I rotate around, all horizontal directions along the surface of the Earth have the same property of not having any force acting along them. The fact that this symmetry allows me to work with an effectively two-dimensional system represents a specific example of a much more general relationship between symmetries and the ease with which I can solve a problem.

Problem 7

The negative solution represents the time at which I could have fired the bullet from the ground behind me, and had it follow the same trajectory.

If I imagine that I were someone standing on the other side of a wall, and got hit with the projectile, I might try to figure out when it was fired. However, there are multiple points along the trajectory from which I could have fired the projectile. If I were to take my gun, and hold it at some position further along the trajectory, and fire it with an initial velocity that the bullet would have otherwise had at that point, then it will continue to finish moving along the rest of the trajectory that it otherwise would have had. This is because once I fire a bullet, its initial position and velocity are enough to completely determine the rest of its motion.

If you really want to convince yourself of this, you can figure out what the negative solution is, and then take our expressions we found for the position and velocity, and evaluate them at that time. If you redo the problem, starting with that time, position, and velocity, you should end up with exactly the same trajectory.

Problem 8

In this problem, we want to find the minimum possible velocity, and also the maximum possible velocity, such that the ball will land in the back of the truck. Before we dive into any math, its helpful to think about *why* there should be a minimum velocity in the first place, and why there should be a maximum velocity. If we think over this question, we realize that a minimum velocity exists because if we fire the projectile too slowly, the truck will move away from the ball too quickly, and the ball will not be able to catch up with the back of the truck. The minimum velocity therefore corresponds to the limiting case in which the ball just barely grazes the back of the truck as it lands inside. In the opposite extreme, if we fire the ball too quickly, it will overshoot the truck, landing on the other side before the truck has reached that point. The maximum velocity in this case clearly corresponds to the situation in which the ball lands at the very front of the trunk, where it would smash into the back window if it were moving any faster.

Let's now explore the motion of the ball in a slightly more quantitative fashion. In the case in which the ball just barely makes it into the back of the trunk, the total horizontal distance that it will travel before landing is

$$d = d_0 + VT,$$

where V is the speed of the truck, and T is the total time of flight. This is just the statement that in order to land in the back of the truck after some total time T, the ball must move a horizontal distance which corresponds to the initial distance from the truck, d_0, plus the extra distance the truck has travelled, which is VT. We also know that the total time T the ball will spend in the air during its flight is

$$T = \frac{2v_0 \sin\theta}{g}.$$

We know this by using the results we derived in lecture, for the case $h = 0$. Thus, in the case in which the ball barely makes it into the truck, we have

$$d = d_0 + V\frac{2v_0 \sin\theta}{g}.$$

We now have an equation which involves v_0, the quantity which we would like to solve for, but we can't quite solve it yet, since we need to know exactly what d is. However, we also have an expression for this quantity, in terms of v_0, from our results derived during lecture. This quantity is given according to

$$d = 2\frac{v_0^2 \sin\theta\cos\theta}{g},$$

where we have again set $h = 0$. Equating these two expressions for d,

$$d_0 + V\frac{2v_0 \sin\theta}{g} = 2\frac{v_0^2 \sin\theta\cos\theta}{g},$$

we now have an equation that we can solve for v_0. Rearranging slightly, we have

$$\frac{2\sin\theta\cos\theta}{g}v_0^2 - \frac{2V\sin\theta}{g}v_0 - d_0 = 0,$$

which is a quadratic equality in v_0, which we can solve by using the quadratic equation. Using the quadratic equation, taking the positive solution (since speeds are positive!), and using the specific values listed in the problem statement, we find

$$v_0 \approx 15.824 \text{ m/s}$$

as the *minimum* velocity.

Now, what about the maximum velocity? In concrete mathematical terms, the statement that the ball just barely avoids overshooting the trunk is the statement that it travels a total horizontal distance

$$d = d_0 + L + VT.$$

This is just the mathematical statement that the ball must travel the distance d_0, in addition to the distance the truck has moved, VT, plus the length of the truck bed L. Thus, everything about solving this case is precisely the same as the previous case, with the mere replacement

$$d_0 \to d_0 + L.$$

The equation which we must solve is therefore

$$\frac{2\sin\theta\cos\theta}{g}v_0^2 - \frac{2V\sin\theta}{g}v_0 - d_0 - L = 0,$$

the solution of which we eventually find to be

$$v_0 \approx 17.041 \text{ m/s}$$

as the *maximum* allowed velocity.

Problem 9

(a) Mathematically speaking, this equation tells us that the rate of change is proportional to how many rabbits are present. We might imagine that this roughly accounts for the fact that the more rabbits there are, the more breeding that is taking place, and the more breeding that is taking place, the more rabbits are being born. This might be the simplest possible formula we could assume for the rate of change of the rabbit population. If the constant k is positive, the rabbit population would be increasing in proportion to the number of rabbits. We might be able to imagine some situations in which the constant is negative, although it seems unlikely that any model would predict that the rabbit population should always be decreasing, in proportion to how many rabbits there are (maybe this model would be appropriate for sterile rabbits that are in the process of fighting each other to the death, for some strange reason). Notice in particular that there are many things this simple equation does not account for. There are no limitations due to finite resources, no terms accounting for predators, and so on. While all of these things are indeed important considerations, our model might still be valid if we are considering what happens when we introduce an invasive species to a new area, where it has no natural predators, and while it still has ample resources to consume.

(b) If we rearrange the equation in part a, we find

$$\frac{1}{R}\frac{dR}{dt} = k$$

If we integrate both sides with respect to time, this becomes

$$\int \frac{1}{R}\frac{dR}{dt}dt = \int kdt \;\Rightarrow\; \int \frac{dR}{R} = \int kdt \;\Rightarrow\; \ln R = kt + C$$

We can rearrange this expression by taking the exponential of both sides. Because the exponential of a logarithm simply gives back the argument of the logarithm, we find

$$R = e^{kt+C} = e^C e^{kt}$$

So we see that according to this model, the rabbit population increases exponentially.

(c) According to the information we have about the rabbit population, and the solution we found above,

$$R\left(t = 0\right) = e^C e^{k \cdot 0} = e^C \cdot 1 = R_0$$

Therefore, the additive constant satisfies

$$e^C = R_0 \;\Rightarrow\; C = \ln R_0$$

More importantly, this information lets us write the solution to our differential equation in terms of the more physically relevant quantity R_0,

$$R = R_0 e^{kt}$$

(d) Our solution tells us that the rabbit population will keep increasing exponentially forever. However, our intuition tells us that most animal populations don't keep increasing exponentially forever. Eventually, other factors come into play, such as limited resources, disease, natural predators, and so on.

(e) When R is much smaller than N, we have

$$R \ll N \;\Rightarrow\; \frac{R}{N} \ll 1$$

So we suspect that the term in the parentheses is probably very closely equal to one,

$$1 - \frac{R}{N} \approx 1,$$

since the second term doesn't subtract very much. So our differential equation becomes approximately

$$\frac{dR}{dt} \approx kR,$$

which is the same equation as in part a. Now, when R is much bigger than N, then we have

$$R \gg N \;\Rightarrow\; \frac{R}{N} \gg 1$$

In this case, the 1 in the parentheses is probably pretty unimportant in comparison with the second term,

$$1 - \frac{R}{N} \approx -\frac{R}{N}.$$

In this situation, our differential equation is approximately

$$\frac{dR}{dt} \approx -\frac{k}{N}R^2.$$

When N and R are equal, then the term in parentheses is zero, and so the rate of change of the rabbit population is zero. Notice that when the rabbit population is less than N, the rate of change is positive, so that the population will increase towards N. When the rabbit population is more than N, the rate of change is negative, so that the population will decrease towards N. When the population is equal to N, it will stay there, since the rate of change is zero. In the language of differential equations, we say that the point $R = N$ represents a *stable equilibrium* of the rabbit population.

(f) If we rearrange the differential equation in question, we have

$$\frac{1}{\left(1 - \frac{R}{N}\right) R} \frac{dR}{dt} = k.$$

We now want to perform an integration on both sides with respect to time. Since we know what the rabbit population is at time zero, and we want to know what it is at some later time T, it seems natural that these are probably the times we would want to use for the boundaries of a definite integral. We'll see in a moment that this does indeed help us arrive at the answer we want. Performing this integration, we have

$$\int_0^T \frac{1}{\left(1-\frac{R}{N}\right)R}\frac{dR}{dt}dt = \int_0^T k\ dt = kT.$$

Now, we have to be careful about the boundaries of integration on the left side. If we change variables from t to $R(t)$, then my integration is now between the endpoints $R(0)$ and $R(T)$. Therefore, we find

$$\int_{R(0)}^{R(T)} \frac{dR}{\left(1-\frac{R}{N}\right)R} = kT.$$

If we use an integral table to look up the value of the integral on the left (or use something like Wolfram Alpha, or a TI-89), then we find the integral on the left to be

$$\ln\left(\frac{R}{R-N}\right)\Bigg|_{R(0)}^{R(T)} = kT$$

If we evaluate the anti-derivative between the two endpoints, we find

$$\ln\left(\frac{R(T)}{R(T)-N}\right) - \ln\left(\frac{R(0)}{R(0)-N}\right) = kT.$$

Using the rules of logarithms, we can combine the two terms on the left, to get

$$\ln\left(\frac{R(T)}{R(T)-N}\frac{R(0)-N}{R(0)}\right) = kT.$$

If we take the exponential of both sides of this equation, we find

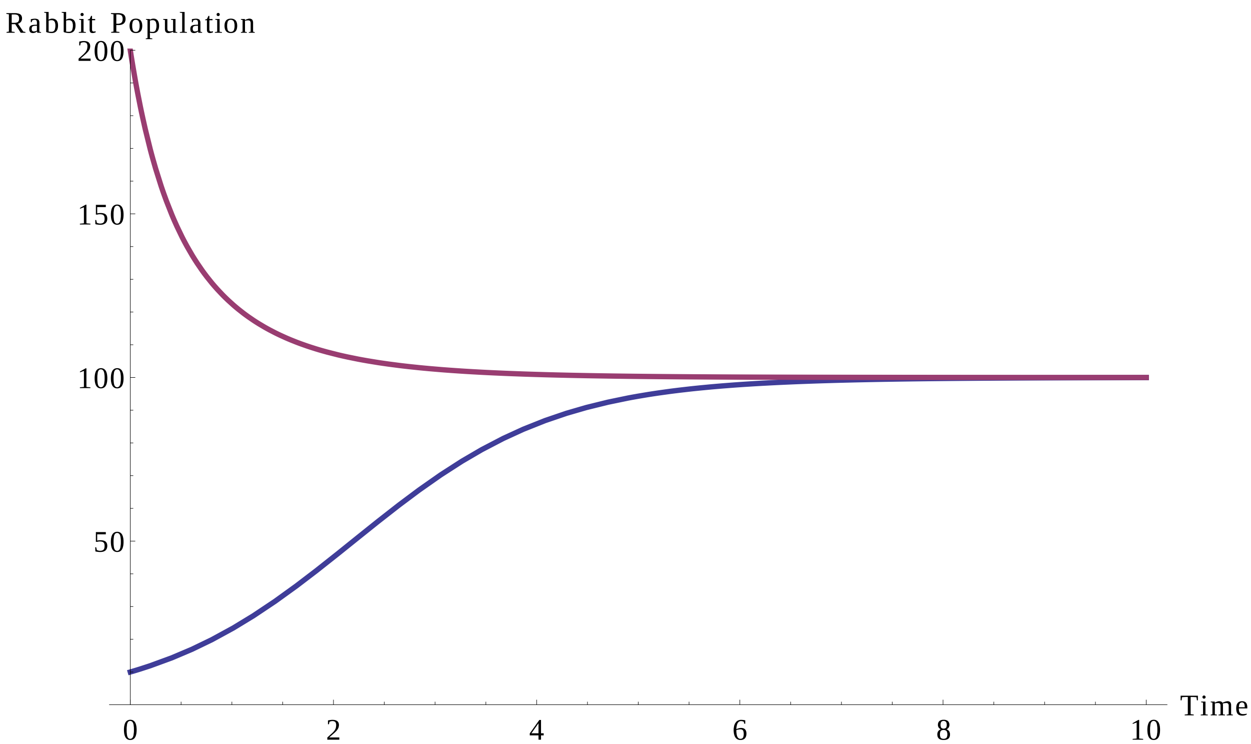
$$\frac{R(T)}{R(T)-N}\frac{R(0)-N}{R(0)} = e^{kT}$$

With a little bit more algebraic rearranging, we finally arrive at the expression

$$R(T) = NR_0\frac{e^{kT}}{R_0e^{kT}+N-R_0}.$$

Notice that because we performed a definite integral and took into account the boundaries to begin with, we didn't have to deal with an intermediate constant C at any point.

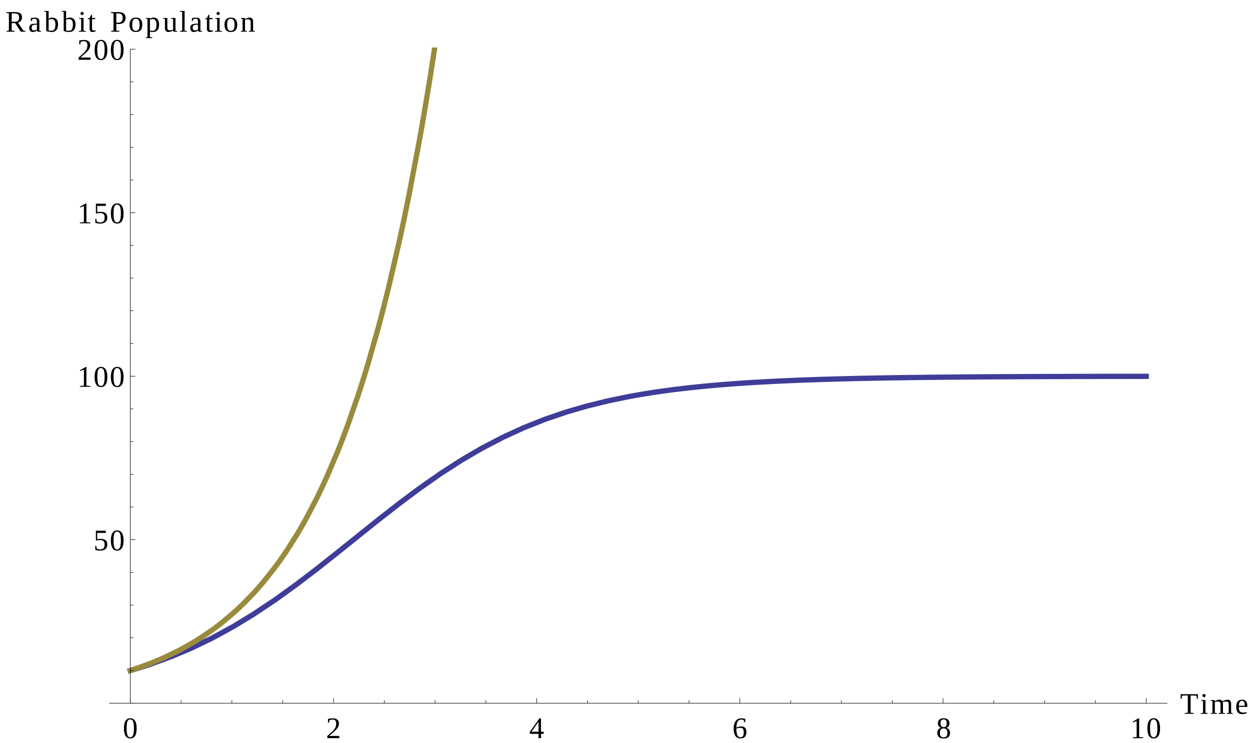
(g) Some plots of the solutions are included below. The blue curve shows the case $R_0 = 10$, while the purple curve shows the case $R_0 = 200$. Notice that both plots tend to converge on the value $N = 100$. For these reason, this parameter is typically referred to as the *carrying capacity* of the population. It reflects the fact that typically, a population will reach a point at which limited resources and environmental factors inhibit further growth. If a population starts out below this value, the population will increase, while if the population starts out above this value, it will decrease.



In the second plot, we've compared what the solution to the $R_0 = 10$ case would be if we had used the approximate form of the differential equation,

$$\frac{dR}{dt} \approx kR.$$

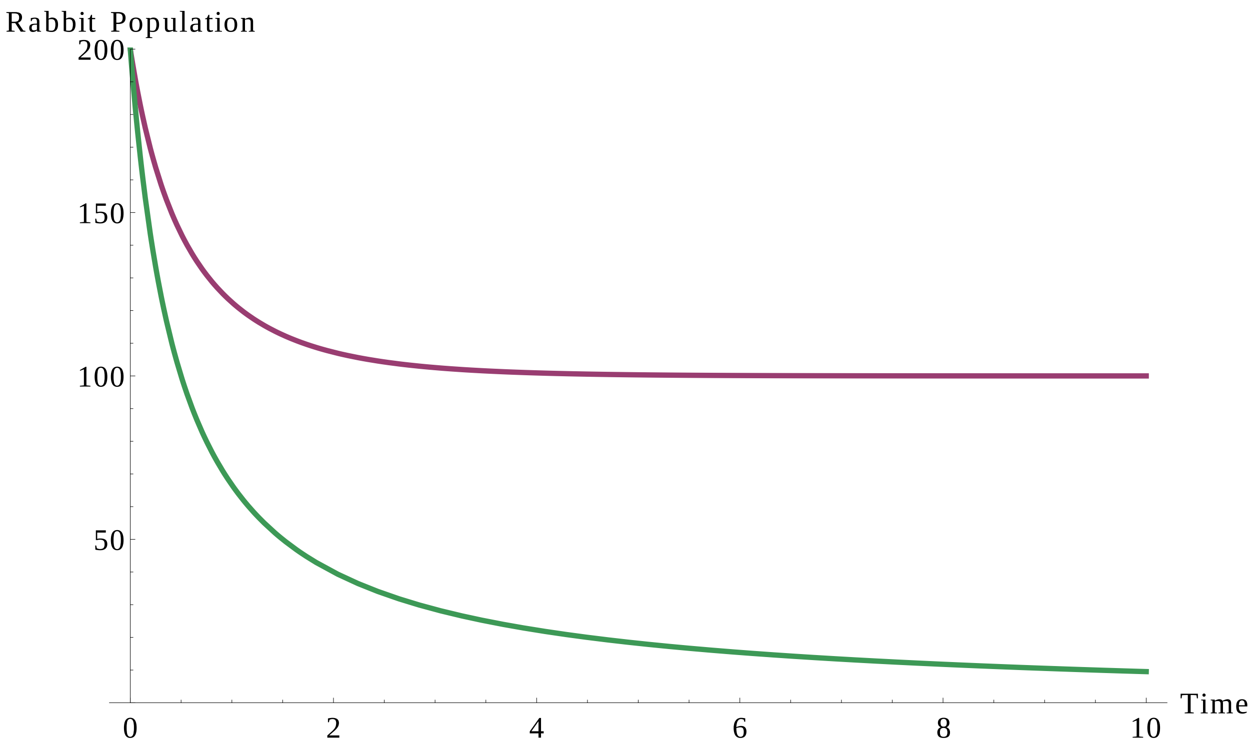
This plot is in gold. Notice that for times close to zero, it agrees reasonably well with the exact solution, but tends to diverge badly at later times. This reflects the fact that well before the population has reached its carrying capacity, it effectively does not feel the limitations of its environment. But this exponential growth cannot continue forever, and eventually the population tapers off to a fixed value. This type of growth is known as *logistic growth*.



We've also included a third plot, which shows the exact solution for the case $R_0 = 200$, along with what the solution would be if we had used the approximate formula,

$$\frac{dR}{dt} \approx -\frac{k}{N} R^2.$$

This plot is shown in green. Even though 200 is not that much larger than 100, the plots still have reasonable agreement for small times. However, they eventually tend to deviate over longer times. Can you derive what the solution to the above approximate equation is?



(h) The second model takes into account the fact that populations typically experience limitations to their growth. The various ways in which these limitations occur can be very complex. They can include diseases, limited resources, seasonal and environmental effects, natural predators, and so on. What this model does is incorporate these effects in the simplest possible way, using a differential equation that gives negative values above the carrying capacity, positive values below the carrying capacity, and zero at the carrying capacity. Of course, we could imagine much more complicated differential equations which reproduce this behaviour, but this model still captures the major qualitative features of limited population growth.