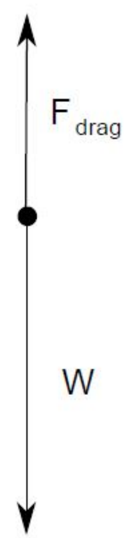
Physics 20 Homework 2
SIMS 2016

Due: Saturday, August 20$^{\text{th}}$

Problem 1

(a) There will be two forces acting on the particle: its weight $\boldsymbol{W}$, which always points down, and the drag force $\boldsymbol{F}_{\text{drag}}$, which can point either up or down, depending on which direction the particle is traveling. For our free body diagram, let's assume the particle is moving downward, so the drag force points upward. Then our diagram looks like this:



(b) Newton's second law states that the net force on an object is equal to its mass times its acceleration. If we choose the positive y-direction to be up, then the net force on the particle is

$$\begin{aligned} F_{\text{net}} &= F_{\text{drag}} - W \\ &= -bv - mg \end{aligned}$$

where I used Stokes' law to write $F_{\text{drag}} = -bv$. But $F_{\text{net}} = ma$, and therefore the acceleration of the particle is

$$\boxed{a = \frac{F_{\text{net}}}{m} = -\frac{bv}{m} - g}$$

A constant velocity implies a zero acceleration; thus the velocity will remain constant when the acceleration is zero. Thus the terminal velocity is defined by

$$a = 0 = -\frac{bv_T}{m} - g$$

$$\boxed{v_T = -\frac{mg}{b}}$$

Note that the terminal velocity is negative: this means that the particle must be falling *down* to achieve terminal velocity (no surprise).

In vacuum, there is no drag force, so $b = 0$. Then we see that the terminal velocity blows up, i.e. an object falling in vacuum will always keep speeding up, so we conclude that there is no terminal velocity in vacuum (if we ignore special relativity, anyway).

(c) Note that from our result above, the terminal velocity of an object falling in a fluid is proportional to m/b. Now, the constant b depends on the size and shape of the falling object, as well as the fluid the object is falling through. If we consider two object with the same size and shape but with very different masses (say, a soap bubble and a ball of lead), the constant b will be the same for both of them. Then the terminal velocity will depend only on m, and so the heavier object will have a greater terminal velocity, and fall faster.

(d) Let's write $a = dv/dt$; then expression for the acceleration from part (b) becomes

$$\frac{dv}{dt} = -\frac{bv}{m} - g$$

To solve this equation, let's rearrange:

$$\frac{dv}{bv/m + g} = -dt$$

Next, let's integrate both sides: we'll integrate the left-hand side from some initial velocity v_0 to a velocity $v(t)$ at time t, and we'll integrate the right-hand side from 0 to t:

$$\int_{v_0}^{v(t)} \frac{dv}{bv/m + g} = -\int_0^t dt' = -t$$

To integrate the left-hand side, let's make the u-substitution $u = bv/m + g$; then $du = (b/m)dv$, or $dv = (m/b)du$, and we get

$$\begin{aligned}
\frac{m}{b} \int_{bv_0/m+g}^{bv(t)/m+g} \frac{du}{u} &= -t \\
[\ln u]_{bv_0/m+g}^{bv(t)/m+g} &= -\frac{bt}{m} \\
\ln(bv(t)/m + g) - \ln(bv_0/m + g) &= -\frac{bt}{m} \\
\ln\left(\frac{bv(t)/m + g}{bv_0/m + g}\right) &= -\frac{bt}{m} \\
\frac{bv(t)/m + g}{bv_0/m + g} &= e^{-bt/m} \\
\frac{bv(t)}{m} + g &= \left(\frac{bv_0}{m} + g\right) e^{-bt/m} \\
v(t) &= -\frac{mg}{b} + \left(v_0 + \frac{mg}{b}\right) e^{-bt/m}
\end{aligned}$$

We recognize the quantity $-mg/b$ as the terminal velocity v_T, so we have

$$\boxed{v(t) = v_T + (v_0 - v_T)\, e^{-bt/m}}$$

In the limit of very late time, i.e. $t \to \infty$, the exponential factor goes to zero, and we get

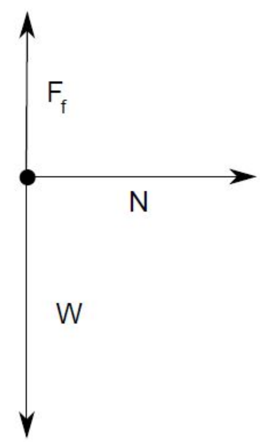
$$v(t \to \infty) = v_T + 0 = v_T$$

so at large times, the velocity will *always* approach the terminal velocity, no matter what the initial velocity v_0 was.

Problem 2

(a) Because the cart is accelerating towards the block, the block must accelerate with the cart. In order to do so, by Newton's second law there must be a net horizontal force acting on the block; this horizontal force is provided by the normal force of the cart on the block. The block would like to fall down, but static friction opposes this motion; since the maximum force of static friction is proportional to the normal force, and the normal force is proportional to the acceleration of the cart, a greater acceleration implies a greater maximum force of static friction. If the acceleration is great enough, the maximum force of static friction should exceed the weight of the block, and the block will not fall.

(b) Below is a free body diagram with all the forces on the block labeled: its weight $\boldsymbol{W}$, the force of static friction $\boldsymbol{F}_f$, and the normal force $\boldsymbol{N}$.



We need to apply Newton's second law twice, once per direction. In the x-direction, we have

$$F_{\text{net},x} = ma_x$$

The net force in the x-direction is just the normal force N, and the acceleration in the x-direction is a, so we have

$$N = ma$$

Likewise, in the y-direction we have

$$F_{\text{net},y} = ma_y$$

The net force in the y-direction is $F_f - W = F_f - mg$, and we want the acceleration in the y-direction to be zero (i.e. we want the block to not fall), so by Newton's second law, we require the force of static friction to satisfy

$$F_f - mg = 0 \Rightarrow F_f = mg$$

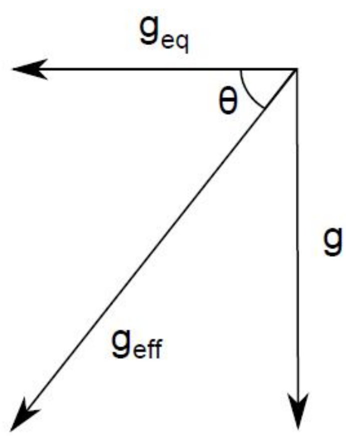
Now, recall that the force of static friction satisfies the inequality $F_f \leq \mu_s N$. From our expression for the normal force, this means that $F_f \leq \mu_s ma$. But in order for the block to not slip, we required $F_f = mg$; thus the block will not slip if the maximum force of static friction, $\mu_s ma$, is greater than the required force of static friction, mg:

$$\mu_s ma \geq mg$$

$$\boxed{a \geq \frac{g}{\mu_s}}$$

(c) (i) The gravitational field that corresponds to a uniform acceleration $\boldsymbol{a}$ has the same magnitude but points in the opposite direction to the acceleration; thus the equivalent gravitational field $\boldsymbol{g}_{\text{eq}}$ has magnitude a and points to the left.

(ii) The usual gravitational field $\boldsymbol{g}$ points down; thus our two gravitational fields form a right triangle, as shown:



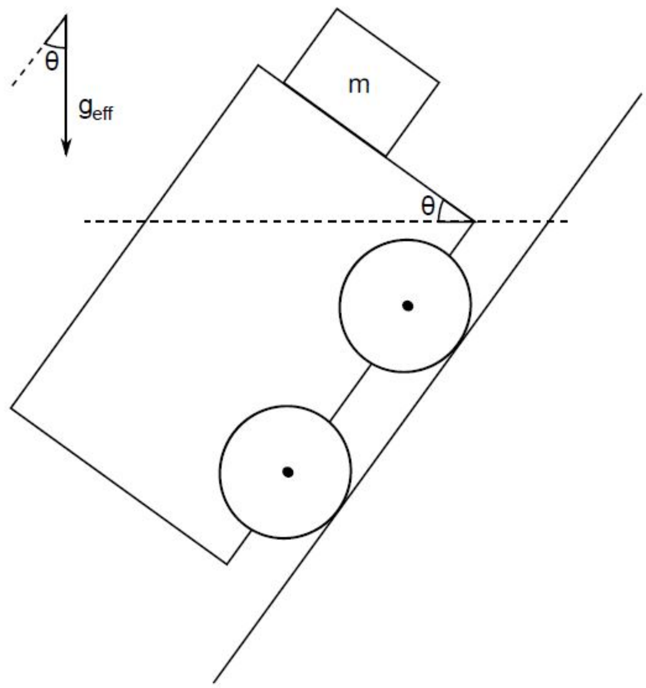
The effective gravitational field therefore has magnitude

$$g_{\text{eff}} = \sqrt{g^2 + g_{\text{eq}}^2} = \sqrt{g^2 + a^2}$$

and the angle it makes with the x-axis is given by $\tan\theta = g/g_{\text{eq}} = g/a$, so

$$\boxed{\theta = \arctan(g/a)}$$

(iii) Note that if we rotate our coordinate system so that $\boldsymbol{g}_{\text{eff}}$ points "down," the front of the cart effectively becomes an inclined plane, as shown:



Now we can completely forget about the accelerating cart, and only work with an inclined plane problem! In particular, we know from class that the block will not slide when $\tan\theta \leq \mu_s$. We found above that $\tan\theta = g/a$, so in order for the block not to slide, we require

$$\tan\theta \leq \mu_s \Rightarrow \frac{g}{a} \leq \mu_s$$

$$\boxed{a \geq \frac{g}{\mu_s}}$$

in agreement with our result from part (b). Thus we get the same answer in a completely different way! In this particular problem, we had to do about the same amount of work with either method, but there are times when you can obtain a result with the equivalence principle with far less work than other methods, and gain some insight into the system to boot.

Problem 3

(a) The key observation here is that relative to the ground, the velocity of the boat will be the velocity of the boat as it would be in still water, plus the velocity of the water. If we choose the x-axis to point downstream and the y-axis to point across the river, and if the boat faces an angle θ from the y-axis, the components of its velocity relative to the water will be

$$v_{BWx} = -v_{BW} \sin\theta$$
$$v_{BWy} = v_{BW} \cos\theta$$

To get the components of the boat's velocity relative to the ground, we simply add the velocity of the water relative to the ground:

$$v_{BGx} = -v_{BW} \sin\theta + v_{WG}$$
$$v_{BGy} = v_{BW} \cos\theta$$

Now, in order for the boat to travel straight across the water, its x-velocity relative to the ground must be zero, so we require

$$v_{BGx} = -v_{BW} \sin\theta + v_{WG} = 0$$

or

$$v_{BW} \sin\theta = v_{WG} \Rightarrow \sin\theta = \frac{v_{WG}}{v_{BW}}$$

(the angle θ exists, since $v_{BW} > v_{WG}$ and therefore $v_{WG}/v_{BW} < 1$). The boat should therefore travel at an angle

$$\boxed{\theta = \arcsin\left(\frac{v_{WG}}{v_{BW}}\right)}$$

upstream.

(b) The y-component of the boat's velocity is

$$v_{BGy} = v_{BW} \cos\theta$$

Using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we can write $\cos\theta = \sqrt{1 - \sin^2\theta}$, and using our result from part (a), we get

$$\cos\theta = \sqrt{1 - \left(\frac{v_{WG}}{v_{BW}}\right)^2}$$

and thus the y-component of the boat's velocity is

$$v_{BGy} = \sqrt{v_{BW}^2 - v_{WG}^2}$$

If the width of the river is d, then the time it takes the boat to cross the river is

$$t = \frac{d}{v_{BGy}}$$

Plugging in our result for v_{BGy}, we get

$$\boxed{t = \frac{d}{\sqrt{v_{BW}^2 - v_{WG}^2}}}$$

Note that as v_{WG} gets closer to v_{BW}, the time it takes for the boat to cross becomes infinite; this is because if the speed of the river equals the speed of the boat in still water, then in order to not be carried downstream, the boat must face directly upstream, and the component of its velocity across the river vanishes.

(c) As before, if the boat faces an angle θ upstream from the y-axis, its velocity relative to the ground will be

$$\begin{aligned} v_{BGx} &= -v_{BW}\sin\theta + v_{WG} \\ v_{BGy} &= v_{BW}\cos\theta \end{aligned}$$

Now, the angle ϕ that the boat's velocity must make to go from point A to point B is given by

$$\tan\phi = \frac{x}{d}$$

But we also have that

$$\tan\phi = \frac{v_{BGx}}{v_{BGy}}$$

Combining these results, we require that

$$\frac{x}{d} = \frac{v_{WG} - v_{BW}\sin\theta}{v_{BW}\cos\theta}$$

or

$$v_{BW}\left(\frac{x}{d}\cos\theta + \sin\theta\right) = v_{WG}$$

Now, using the formula given in the hint, we can write

$$\frac{x}{d}\cos\theta + \sin\theta = \sqrt{\left(\frac{x}{d}\right)^2 + 1}\,\sin(\theta + \delta)$$

where $\tan\delta = x/d$. We then get

$$v_{BW}\sqrt{\left(\frac{x}{d}\right)^2 + 1}\,\sin(\theta + \delta) = v_{WG}$$

or

$$\sin(\theta + \delta) = \frac{v_{WG}/v_{BW}}{\sqrt{(x/d)^2 + 1}} \Rightarrow \theta = \arcsin\left(\frac{v_{WG}/v_{BW}}{\sqrt{(x/d)^2 + 1}}\right) - \delta$$

The angle the boat should face is therefore

$$\boxed{\theta = \arcsin\left(\frac{v_{WG}/v_{BW}}{\sqrt{(x/d)^2+1}}\right) - \arctan(x/d)}$$

Note that when $x = 0$ (i.e. when the boat needs to travel directly across the river), we end up with $\theta = \arcsin(v_{WG}/v_{BW})$, in agreement with our result from part (a). What about the limiting cases? Well, if point B were very far downstream, i.e. if $x/d \to \infty$, then

$$\arcsin\left(\frac{v_{WG}/v_{BW}}{\sqrt{(x/d)^2+1}}\right) \to 0$$
$$\arctan(x/d) \to \frac{\pi}{2}$$

so $\theta \to -\pi/2$, which means the boat would need to face straight downriver - no surprise!

(d) Now, in order for the angle θ to exist, the argument of the inverse sine must be less than 1 (because the domain of the inverse sine is $[-1, 1]$). This means that in order for the boat to be able to reach the ground at point B, we must have

$$\frac{v_{WG}/v_{BW}}{\sqrt{(x/d)^2+1}} \leq 1$$

or

$$\boxed{v_{BW} \geq \frac{v_{WG}}{\sqrt{(x/d)^2+1}}}$$

When $x = 0$, the minimum speed the boat can have to reach the opposite ground is v_{WG}, as we assumed in part (a) above. As x increases, the boat can reach point B with lower and lower speeds, until we send $x \to \infty$, in which case the boat can reach point B even if v_{BW} is zero (in this case, the river simply carries the boat all the way downstream).

Problem 4

(a) We know that the position vector, in terms of the angle and radius, can always be written as

$$\boldsymbol{r} = r\cos\theta\hat{x} + r\sin\theta\hat{y},$$

so using the definition of the unit vector $\hat{r}$, we have

$$\hat{r} = \frac{\boldsymbol{r}}{r} = \cos\theta\hat{x} + \sin\theta\hat{y}.$$

Now, the vector $\hat{\theta}$ is defined to be a ninety degree rotation counter-clockwise from $\hat{r}$, which is simply

$$\hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y}.$$

In order to check that this is the correct choice of signs, consider the limiting cases! Does this expression make sense when $\theta = 0$?

(b) This one is simple - based on the definition of $\hat{r}$, we merely have

$$\hat{r} = \frac{\boldsymbol{r}}{r} \Rightarrow \boldsymbol{r} = r\hat{r}.$$

There is of course no component along $\hat{\theta}$.

(c) We know that the position vector, in terms of r and θ, can be written as

$$\boldsymbol{r} = r\cos\theta\hat{x} + r\sin\theta\hat{y}.$$

Therefore, computing the velocity in the original Cartesian basis, we find

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}\left(r\cos\theta\right)\hat{x} + \frac{d}{dt}\left(r\sin\theta\right)\hat{y},$$

or, making use of the product rule (since r and θ both depend on time!),

$$\boldsymbol{v} = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\hat{x} + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\hat{y}.$$

If we rearrange this equation slightly, we can write

$$\boldsymbol{v} = \dot{r}\left(\cos\theta\hat{x} + \sin\theta\hat{y}\right) + r\dot{\theta}\left(-\sin\theta\hat{x} + \cos\theta\hat{y}\right).$$

Now, comparing with our expressions from part a, we see that this expression for the velocity can also be written as

$$\boldsymbol{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}.$$

(d) Examining our results from the previous sections, we found that

$$g_r\left(r, \theta, \dot{r}, \dot{\theta}\right) = \dot{r},$$

whereas

$$\frac{d}{dt} f_r\left(r, \theta\right) = \frac{d}{dt}\left(r\right) = \dot{r}.$$

Thus, for the r-component, both expressions agree. Physically, this result tells us that the rate of change of position, along the direction of the position vector, is the rate of change of the length of the vector.

As for the θ component, we found above that

$$g_\theta\left(r, \theta, \dot{r}, \dot{\theta}\right) = r\dot{\theta},$$

whereas

$$\frac{d}{dt} f_\theta\left(r, \theta\right) = \frac{d}{dt}\left(0\right) = 0.$$

These expressions are certainly not equal! At least, not as long as r and $\dot{\theta}$ are non-zero. Physically, this result tells us that the rate of change of the position vector, in a direction perpendicular to the rate of change of radius, is dependent on both the radius itself, and the rate of change of the angular coordinate. The rate of change in this direction gives a sense of how much the particle's trajectory is being rotated around the origin, and because the basis vectors in this case depend on time and are not fixed in space, the simple expressions we derived in lecture for the Cartesian basis vectors no longer apply.

Problem 5

(a) At the instant I let the bodies go, their separation is r, and I can use the law of gravitation to compute the magnitude of the force on each object. I can also use this law to compute the magnitude of the acceleration of each object. For body one, I have

$$a_1 = \frac{F_1}{m_1} = \frac{F_g}{m_1} = G\frac{m_1 m_2}{m_1 r^2} = G\frac{m_2}{r^2}$$

So we see that the magnitude of the acceleration of body one, at the instant I let it go, does not depend on its own mass. The direction of the acceleration also does not depend on its mass, since the direction is defined to point towards the second body. So then in total, the acceleration of the first body does not depend on its own mass. By going through the same calculation for the second body, we can see that its acceleration does not depend on its own mass either.

(b) Newton's law of gravitation states that the magnitude of the force on each body is given by

$$F_g = G\frac{m_1 m_2}{r^2},$$

and so in agreement with Newton's third law, the magnitudes of the forces on each body are always the same. However, the accelerations will not be the same. For body one, the magnitude of the acceleration is

$$a_1 = G\frac{m_2}{r^2},$$

while for body two we have

$$a_2 = G\frac{m_1}{r^2}.$$

If the first body is much more massive, then we have

$$m_1 \gg m_2 \Rightarrow \frac{G}{r^2} m_1 \gg \frac{G}{r^2} m_2 \Rightarrow a_2 \gg a_1.$$

So the lighter body will experience a much larger acceleration, as we might intuitively expect.

(c) Even though the forces and accelerations will be changing, Newton's law of gravitation still gives the force, and thus acceleration, in terms of the distance between the two objects. If the distance between them is changing with time, it is still true that for the first body,

$$a_1(t) = G\frac{m_2}{r^2(t)},$$

while for the second body

$$a_2(t) = G\frac{m_1}{r^2(t)}.$$

In general, this presents a differential equation we need to solve. But, if we don't wait long enough for the distance to start changing a lot, then the acceleration over the course of time doesn't change too much either. For example, if I let go of a ball on the Earth, as it moves down and hits the ground, its distance from the center of the Earth (which is the important distance for spherical bodies) barely changes at all, and so over the course of motion, the accelerations don't change too much.

Because of this, the conclusion I made at the instant I let them go is still true - the acceleration of the larger body is much, much smaller. Because this acceleration is what causes the body to change its velocity, then certainly, this body will not move very much at all, since its acceleration is so tiny. The lighter body, on the other hand, has a huge acceleration, comparatively, and so it will move a lot. Of course, this idea is familiar to us - when I let go of a ball, despite the fact that the two bodies attract each other, the ball certainly accelerates a lot more than the Earth! I don't see the Earth rush up to meet the ball.

(d) To be specific, I'll set up a coordinate system where the positive y direction points *away from the surface of the Earth.* Then Newton's law of gravitation tells me that the force on the ball is given by

$$\boldsymbol{F} = -G\frac{Mm}{r^2}\hat{y},$$

where r is the distance between the center of the Earth and the center of the ball. In terms of the radius of the Earth, and the height of the ball above the ground, I can write this as

$$\boldsymbol{F} = -G\frac{Mm}{(R+h)^2}\hat{y}.$$

The acceleration is given by Newton's second law, and I find

$$\boldsymbol{a} = -G\frac{M}{(R+h)^2}\hat{y}.$$

(e) When $h = 0$, we find

$$\boldsymbol{a} = -G\frac{M}{R^2}\hat{y}.$$

This is a vector which is oriented towards the ground. Now, if we use numerical values for G, M, and R, we find that the magnitude of the acceleration is

$$\frac{GM}{R^2} \approx 9.8\ \frac{\mathrm{m}}{\mathrm{s}^2},$$

the usual (constant) acceleration due to gravity! So this number isn't just some magic numerical value that comes from nowhere - we can understand how it arises from a more fundamental law as just an approximation over small distances.

(f) If we use the approximation formula we are given, we can rewrite the acceleration from part d as

$$\boldsymbol{a} = -G\frac{M}{(R+h)^2}\hat{y} \approx -GM\left[\frac{1}{R^2} - 2\frac{h}{h^3}\right]\hat{y} = -\frac{GM}{R^2}\left[1 - 2\frac{h}{R}\right]\hat{y}$$

The overall prefactor is the usual result, whereas the term in parentheses is an overall multiplicative factor that is very close to one (notice that the units work out!). And by close to one, I mean REALLY close to one. The radius of the Earth is about 6.4 thousand kilometers, whereas I often consider dropping a ball over distances of about one meter. So in this situation, we would have

$$1 - 2\frac{h}{R} = 1 - \frac{2}{6.4 \times 10^6} \approx 0.9999997.$$

Thus, the first order correction is around 99.99997 percent of the value we usually use. Needless to say, it seems like we are usually making a pretty good approximation.

(g) The approximation formula given in the previous part is a result of Taylor's theorem, which says that (most) functions can be approximated by a formula which involves their derivatives,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

A discussion of this subject is included in my notes on Taylor series, which are posted online. If a is some constant parameter, and we take b to be some variable which is close to zero, then Taylor's theorem tells us

$$f(b) = \frac{1}{(a+b)^2} \approx f(b=0) + \frac{df}{db}(b=0).$$

Computing the first derivative, we find

$$\frac{df}{db} = \frac{d}{db}(a+b)^{-2} = -\frac{2}{(a+b)^3},$$

which yields

$$f(b) = \frac{1}{(a+b)^2} \approx \frac{1}{a^2} - 2\frac{b}{a^3}$$

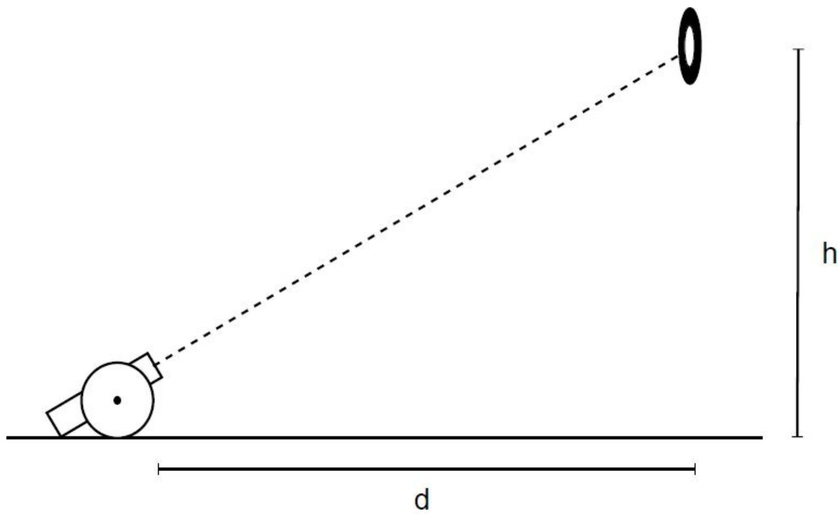
(h) If we denote the height above the ground as y, then we find a differential equation for the acceleration,

$$\ddot{y} = -\frac{GM}{R^2}\left[1 - 2\frac{y}{R}\right]$$

This is a second order differential equation which can be solved using methods which I am happy to explain during office hours to those who are curious.

Problem 6

For convenience, I've included the setup of problem four again. Now, the gravitational equivalence principle tells us that while we actually have a gravitational field in our laboratory, we can always pretend that the laboratory is actually some rocket out in empty space which is accelerating. So I now pretend that I'm in such a rocket, and that the cannon is sitting on the floor of the rocket, while the target hangs from the ceiling. Both of them will be accelerated along with the rocket.



However, imagine that at time $t = 0$, I set up an inertial reference frame where the rocket is instantaneously at rest. In other words, I imagine that at time zero, the rocket is beginning to accelerate, and at this precise moment, I let go of the target, and fire the bullet. While it is true that the floor of the rocket, and hence the cannon, will accelerate upwards, the target and bullet are now out in empty space, not touching anything or feeling any forces. So the target will simply sit still, while the bullet moves along a straight line, directly at the target. Clearly, the bullet will hit the target.

Now, despite the fact that I told you not to do any math, in addition to arriving at this conclusion, I could also say something quantitative. Notice that in my reference frame, the bullet travels at constant speed straight at the target. So the time it will take before hitting the target is simply

$$t_{\text{hit}} = \frac{\text{distance}}{\text{speed}} = \frac{\sqrt{d^2 + h^2}}{v},$$

which is exactly the same conclusion as before.