

Maximally entangling tripartite protocols for Josephson phase qubits

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We introduce a suit of simple entangling protocols for generating tripartite GHZ and W states in systems with anisotropic exchange interaction $g(XX + YY) + \tilde{g}ZZ$. An interesting example is provided by macroscopic entanglement in Josephson phase qubits with capacitive ($\tilde{g} = 0$) and inductive ($0 < |\tilde{g}/g| < 0.1$) couplings.

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I. INTRODUCTION

Superconducting circuits with Josephson junctions have attracted considerable attention as promising candidates for scalable solid-state quantum computing architectures. The story began in the early 1980's, when Tony Leggett made a remarkable prediction that under certain experimental conditions the macroscopic variables describing such circuits could exhibit a characteristically quantum behavior [1]. Several years later such behavior was unambiguously observed in a series of tunneling experiments by Devoret et al. [2], Martinis et al. [3], and Clarke et al. [4]. It was eventually realized that due to their intrinsic anharmonicity, the ease of manipulation, and relatively long coherence times [5], the metastable macroscopic quantum states of the junctions could be used as the states of the qubits. That idea had recently been supported by successful experimental demonstrations of Rabi oscillations [6], high-fidelity state preparation and measurement [7, 8, 9, 10, 11, 12], and various logic gate operations [8, 9, 10, 11, 13]. Further progress in developing a workable quantum computer will depend on the architecture's ability to generate various multiqubit entangled states that form the basis for many important information processing algorithms [14].

In this paper we develop several *single-step* entangling protocols suitable for generating maximally entangled quantum states in tripartite systems with pair-wise coupling $g(XX + YY) + \tilde{g}ZZ$. We base our approach on the idea that implementing symmetric states may conveniently be done by symmetrical control of all the qubits in the system. This bears a resemblance to approaches routinely used in digital electronics: while an arbitrary gate (for example, a 3-bit gate) can be made from a collection of NAND gates, it is often convenient to use more complicated designs with three input logic gates to make the needed gate faster and/or smaller.

The protocols developed in this paper may be directly applied to virtually any of the currently known superconducting qubit architectures, two of which will be mentioned here. The first architecture is based on capacitively coupled current-biased (CBJJ) Josephson junctions [12, 13, 15] whose dynamics is governed by the circuit Hamiltonian

$$H_1 = (p_1^2 + p_2^2 + \kappa p_1 p_2) / 2m + (\hbar/2e) \sum_{i=1}^2 [-I_0 \cos \phi_i - I_i \phi_i], \quad (1)$$

with $p_i = m\dot{\phi}_i$, $m = (\hbar/2e)^2 (C + C_{\text{int}})$, $\kappa = 2C_{\text{int}}/(C + C_{\text{int}})$. The other architecture involves inductively coupled flux-biased (FBJJ) junctions [16]. It is described by the Hamiltonian (for small Υ , see Ref. [17] for details)

$$H_2 = (p_1^2 + p_2^2) / 2m + (\hbar/2e) \sum_{i=1}^2 \left[-I_0 \cos \phi_i + (eE_0/\hbar) (\phi_i - 2\pi\Phi_i/\Phi_{\text{sc}})^2 \right] + \Upsilon E_0 (\phi_1 - 2\pi\Phi_i/\Phi_{\text{sc}}) (\phi_2 - 2\pi\Phi_i/\Phi_{\text{sc}}), \quad (2)$$

with $p_i = m\dot{\phi}_i$, $m = (\hbar/2e)^2 C$, $\omega_0 = 1/\sqrt{LC}$, $E_0 = \hbar^2 \omega_0^2 / 2E_C$, $E_C = (2e)^2 / 2C$, $\Phi_{\text{sc}} = \hbar/2e$, $\Upsilon = M/L$. When reduced to computational subspace, in the rotating wave approximation, these Hamiltonians become

$$H_{\text{RWA}} = (1/2) \left[\vec{\Omega}_1 \cdot \vec{\sigma}_1 + \vec{\Omega}_2 \cdot \vec{\sigma}_2 + g (\sigma_x^1 \sigma_x^2 + \sigma_y^1 \sigma_y^2) + \tilde{g} \sigma_z^1 \sigma_z^2 \right], \quad (3)$$

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with $\tilde{g} = 0$ (momentum-momentum coupling) and $0 < \tilde{g}/g < 0.1$ [18] (position-position coupling), respectively. For typical superconducting qubits, the RWA requirements are well satisfied: the level splittings are usually around $\omega/2\pi \simeq 10$ GHz and the Rabi frequencies Ω needed to implement various logic gates [17, 19] are on the order of the coupling constant $g/2\pi \lesssim 100$ MHz. Thus, $\Omega/\omega \sim 10^{-2}$, as required.

II. THE GHZ PROTOCOL

A. Triangular coupling scheme

In the rotating frame in the absence of coupling, the computational basis states $|000\rangle, |001\rangle, |010\rangle, |100\rangle, |011\rangle, |101\rangle, |110\rangle, |111\rangle$ have the same effective energy $E_{\text{eff}} = 0$. The pair-wise coupling,

$$H_{\text{int}} = (1/2) \sum_{i,j=1}^3 g (\sigma_x^i \sigma_x^j + \sigma_y^i \sigma_y^j) + \tilde{g} \sigma_z^i \sigma_z^j = \begin{pmatrix} 3\tilde{g}/2 & & & & & & & \\ & -\tilde{g}/2 & g & g & & & & \\ & g & -\tilde{g}/2 & g & & & & \\ & g & g & -\tilde{g}/2 & & & & \\ & & & & -\tilde{g}/2 & g & g & \\ & & & & g & -\tilde{g}/2 & g & \\ & & & & g & g & -\tilde{g}/2 & \\ & & & & & & & 3\tilde{g}/2 \end{pmatrix}, \quad (4)$$

(empty matrix elements are zero), *partially lifts* the degeneracy, which results in the energy spectrum

$$E_{\text{int}} = \{3\tilde{g}/2, 3\tilde{g}/2, 2g - \tilde{g}/2, 2g - \tilde{g}/2, -(g + \tilde{g}/2), -(g + \tilde{g}/2), -(g + \tilde{g}/2), -(g + \tilde{g}/2)\}, \quad (5)$$

with the corresponding \mathcal{H} -eigenbasis

$$\mathcal{H}_{\text{GHZ}} \oplus \mathcal{H}_{\text{W}} \oplus \mathcal{H}_{\text{rest}} \equiv \{|000\rangle \oplus |111\rangle\} \oplus \{|W\rangle \oplus |W'\rangle\} \oplus \{|\Psi_1\rangle \oplus |\Psi_1'\rangle \oplus |\Psi_2\rangle \oplus |\Psi_2'\rangle\}, \quad (6)$$

where

$$\begin{aligned} |W\rangle &= (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}, & |W'\rangle &= (|011\rangle + |101\rangle + |110\rangle)/\sqrt{3}, \\ |\Psi_1\rangle &= (|100\rangle - |010\rangle)/\sqrt{2}, & |\Psi_1'\rangle &= (|011\rangle - |101\rangle)/\sqrt{2}, \\ |\Psi_2\rangle &= (|100\rangle + |010\rangle - 2|001\rangle)/\sqrt{6}, & |\Psi_2'\rangle &= (|011\rangle + |101\rangle - 2|110\rangle)/\sqrt{6}. \end{aligned} \quad (7)$$

Since the coupling does not cause transitions within each of the degenerate subspaces (nor does it cause transitions between different such subspaces), it is impossible to generate the $|\text{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ state [20] from the ground state $|000\rangle$ by direct application of H_{int} . Instead, we must first bring the $|000\rangle$ state out of the \mathcal{H}_{GHZ} subspace by, for example, subjecting it to a local rotation R_1 in such a way as to produce a state $|\psi\rangle$ that has both $|000\rangle$ and $|111\rangle$ components. That is only possible if *all* one-qubit amplitudes α_1, \dots, β_3 in the resulting product state $|\psi\rangle = R_1|000\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle)(\alpha_2|0\rangle + \beta_2|1\rangle)(\alpha_3|0\rangle + \beta_3|1\rangle)$ are chosen to be nonzero, which means that in the computational basis the state $|\psi\rangle$ will have eight nonzero components.

We now notice that in the \mathcal{H} -basis, the three-qubit rotations

$$\begin{aligned} X_\theta &= X_\theta^{(3)} X_\theta^{(2)} X_\theta^{(1)} = \begin{pmatrix} c^3 & is^3 & -i\sqrt{3}sc^2 & -\sqrt{3}cs^2 \\ is^3 & c^3 & -\sqrt{3}cs^2 & -i\sqrt{3}sc^2 \\ -i\sqrt{3}sc^2 & -\sqrt{3}cs^2 & c(1-3s^2) & is(1-3c^2) \\ -\sqrt{3}cs^2 & -i\sqrt{3}sc^2 & is(1-3c^2) & c(1-3s^2) \end{pmatrix} \oplus \begin{pmatrix} c & is \\ is & c \end{pmatrix} \oplus \begin{pmatrix} c & is \\ is & c \end{pmatrix}, \\ Y_\theta &= Y_\theta^{(3)} Y_\theta^{(2)} Y_\theta^{(1)} = \begin{pmatrix} c^3 & -s^3 & -\sqrt{3}sc^2 & \sqrt{3}cs^2 \\ s^3 & c^3 & \sqrt{3}cs^2 & \sqrt{3}sc^2 \\ \sqrt{3}sc^2 & \sqrt{3}cs^2 & c(1-3s^2) & s(1-3c^2) \\ \sqrt{3}cs^2 & -\sqrt{3}sc^2 & -s(1-3c^2) & c(1-3s^2) \end{pmatrix} \oplus \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \oplus \begin{pmatrix} c & s \\ -s & c \end{pmatrix}, \end{aligned} \quad (8)$$

where $Y_\theta^{(k)} = \exp(-i\theta\sigma_y^k/2)$, $X_\theta^{(k)} = \exp(-i\theta\sigma_x^k/2)$, $k = 1, 2, 3$, are block-diagonal, with $c \equiv \cos(\theta/2)$ and $s \equiv \sin(\theta/2)$. For $\theta = \pi/2$, the corresponding 4×4 blocks acting on the $\mathcal{H}_{\text{GHZ}} \oplus \mathcal{H}_{\text{W}}$ subspace are

$$X_{\pi/2}^{(4 \times 4)} = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & i & -i\sqrt{3} & -\sqrt{3} \\ i & 1 & -\sqrt{3} & -i\sqrt{3} \\ -i\sqrt{3} & -\sqrt{3} & -1 & -i \\ -\sqrt{3} & -i\sqrt{3} & -i & -1 \end{pmatrix}, \quad Y_{\pi/2}^{(4 \times 4)} = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & -1 & -\sqrt{3} & \sqrt{3} \\ 1 & 1 & \sqrt{3} & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & -1 & -1 \\ \sqrt{3} & -\sqrt{3} & 1 & -1 \end{pmatrix}. \quad (9)$$

IV. ADDENDUM: ISOTROPIC HEISENBERG EXCHANGE $g(XX + YY + ZZ)$

Maximally entangling protocols introduced in previous sections are singular in the limit $\tilde{g} \rightarrow g$, which corresponds to the isotropic Heisenberg exchange interaction. Even though this limit is not met in superconducting qubits, for completeness, we briefly discuss it here.

It is obvious that when $g = \tilde{g}$, the symmetric state $Y_{\pi/2}|000\rangle$ is an eigenstate of the interaction Hamiltonian. Consequently, the Heisenberg exchange does not cause transitions out of it, making the gate time divergent. To perform single-step entanglement we break the symmetry of local rotations. For example, the GHZ state can be generated by $e^{-i\alpha}|\text{GHZ}\rangle = e^{-i(\pi/2)\sigma_z^2}e^{-i(\pi/3)(\sigma_y^1 - \sigma_y^2)}U_{\text{int}}e^{-i(\pi/12)(5\sigma_y^1 + \sigma_y^2 - 3\sigma_y^3)}e^{-i(\pi/2)\sigma_z^2}|000\rangle$, with $\alpha = -\pi/2$, $t_{\text{GHZ}} = (2/3) \times (\pi/2g)$. To generate the W state, we generalize Neeley's fast implementation for triangular $g(XX + YY)$ coupling [23] (*cf.* [24]) to arbitrary coupling $g(XX + YY) + \tilde{g}ZZ$, including the Heisenberg exchange $g = \tilde{g}$: $e^{-i\alpha}|\text{W}\rangle = e^{+i(\pi/3)\sigma_z^2}U_{\text{int}}e^{-i(\pi/2)\sigma_y^2}|000\rangle$, with $\alpha = (5g - 2\tilde{g})\pi/18g$, $t_{\text{W}} = (4/9) \times (\pi/2g)$.

V. CONCLUSION

In summary, we have developed several single-step *symmetric* implementations for generating maximally entangled tripartite quantum states in systems with anisotropic exchange interaction, which are directly applicable to superconducting qubit architectures. In the GHZ case, both triangular and linear coupling schemes have been analyzed. In the isotropic limit, our implementations exhibit singularities that can be removed by breaking the symmetry of the local pulses.

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