Forces and Newton's Laws

Aug. 16, 2017

In the last lecture, we derived the equations governing constantly accelerating motion. Why did we stop at constant acceleration? It's because this is related to a constant force. How is it related to force? Like this:

$$\vec{F} = m\vec{a}. \tag{1}$$

The quantity m appearing in this equation is the mass, or *inertial mass*. The distinction is not important, but for a time there was a debate over whether or not this mass was equivalent to the *gravitational mass* appearing in the famous law of gravitation between two bodies,

$$a_g = \frac{GMm}{r^2} \tag{2}$$

where M and m are the gravitational masses between the two bodies. The fact that the two are related helped Einstein realize that there is nothing special about gravity. It is just the curvature of spacetime and as such the gravitational mass must be related to the inertial mass. In a sense, mass is just the proportionality between force and acceleration, though it often does help quantify things–for instance knowing the mass of a chunk of aluminum can tell you how many aluminum atoms there are in the chunk.

Equation 1 is called *Newton's second law*. In the limit of zero force, it reduces to Newton's first law which states that in an inertial reference frame an object moving at constant velocity remains at constant velocity unless acted upon by a force. The third and final of Newton's laws states that when one body exerts a force on a second body, the second body exerts an equal and opposite force on the first body.

There are a few standard forces in Newtonian mechanics that all physics students are required to learn. They are the force of a ramp, the force of a pulley, and the force of gravity. Given these forces, it is surprising that physicsts don't have more broken bones. Despite the bad humor, these examples go a long way into illustrating the utility of Newton's laws and help in defining *free body diagrams*. A free body diagram is a tool to help understand both the direction and magnitude of a sum of forces acting on a body. It is the *net force* which ends up causing motion and so we will become masters of this tool.

Let's do a ramp problem first. These are also called inclined planes for the less radical. Consider the ramp in figure 1. A 10 kg mass sits at the top of this ramp. This mass experiences both static and kinetic friction. The coefficient of static friction is $\mu_s = .4$ and the coefficient of kinetic friction is $\mu_k = .2$.

a) What is the required angle of the ramp so that the block moves?
b) If the ramp is 2 m long, how long does it take for the block to slide down the ramp at twice this angle? Assume that the ramp is infinitely heavy and does not slide.

In this problem, there are a few new concepts. The first is fricition. Friction is a force that always opposes the direction of motion. To find this force, we multiply the normal force by the friction coefficient. In this case, there are two friction forces,

$$F_s = \mu_s F_N \tag{3}$$

$$F_k = \mu_k F_N. \tag{4}$$

In these equations, everything is written in terms of scalars. These are the norms of the respective force vectors.

One force vector you are probably wondering about is $\vec{F}_N$. Recall from Newton's third law that for every force from one body there is an equal and opposite force on that body. When we place the block on the ramp, it is "stuck" to the surface. While it can slide along the surface, it does not pass through the ramp and it similarly does not jump off the ramp. Gravity is keeping it stuck to surface. Gravity exerts a force on the block which in turn exerts a force on the ramp and the third law tells us that there is an equal force in the opposite direction. These are the only forces perpendicular(normal) to the plane of the ramp and they cancel.

The block can slide if it overcomes the static friction force. To find when this occurs, we note that there are also two forces in this direction, gravity and friction. How can gravity act to keep the block stuck to the ramp and make it move? This is because the normal and tangential axes of the ramp are not aligned with gravity. Hence the gravitational force is a vector which has components along both directions. The static friction force obtained in equation 4 is the maximum force the ramp can apply to the block before it slips. The block will slip when the tangential component of the gravitational force exceeds the maximum static friction force. We increase the gravitational force by increasing the slope of the ramp, making it more aligned with gravity. For smaller angles, the static friction force is equal to the tangential component of the gravitational force because there is no acceleration (Newton's first law–no net force equals no acceleration).

For angles above this maximum angle, the box slides and the tangential force is greater than the friction force. Now we must be careful because the friction force is given by the kinetic friction force. The net force divided by the mass gives the acceleration (Newton's second law). It is just a matter of applying the formulae obtained in lecture 2 to find the time it takes for the block to reach the bottom. Let's do it!

To start, we need to pick a reference frame. A particularly convenient reference frame is one in which motion occurs in only one dimension. Let's choose $\hat{x}$ to be tangent to the ramp with positive $\hat{x}$ pointing toward the top of the ramp. Then $\hat{y}$ points in the perpendicular direction away from the ramp.

a) In this reference frame, we can decompose the gravitational force vector as

$$\vec{F}_g = -|\vec{F}_g| \left(\sin\theta\hat{x} + \cos\theta\hat{y}\right)$$

The minus sign is because gravity points in both the $-\hat{x}$ and $-\hat{y}$ directions, but the norm of a vector is always positive. In this case $|\vec{F}_g| = 98N$. We said that the normal force balances the perpendicular gravitational force and so

$$F_{g,y} + F_N = 0 \Rightarrow F_N = F_g \cos\theta.$$

Now we look at forces in the tangential direction. The maximum angle without motion is when the maximum static frictional force and gravity balance

$$\mu_s F_N = F_{g,x} \Rightarrow \mu_s = \tan\theta.$$

Plugging in numbers (something you should always wait until the end of a problem to do) gives

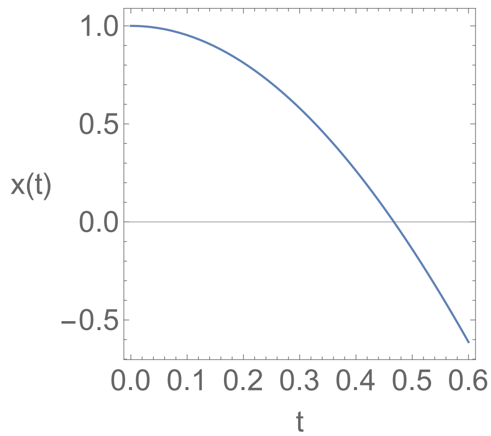
$$\theta = .38 = \pi/8.26.$$

b) Now, to find how long it takes for the block to slide, we use Newton's second law,

$$a_x = \frac{1}{m}\sum F_x = \frac{F_g}{m}(\mu_k \sin\theta - \cos\theta) = g(\mu_k \sin\theta - \cos\theta).$$

Now the time that it takes to travel the length of the plane is

$$t = \sqrt{\frac{2\Delta x}{a_x}} = .59s \tag{5}$$



It is good when doing these problems to have some built in checks. One check is on orders of magnitude. The fastest an object could drop one meter is if it did so vertically with no other force but gravity. It would then take $.45s$. Our block moved slightly slower as expected. Another check is that our final answer does not depend on the mass. This makes sense as the force due to friction (and normal force) are proportional to the gravitational force and we already established that inertial mass and gravitational mass are the same. Hence, they cancel and there is no mass dependence.

Now consider the same ramp as in part (b), but attached to the block is a rope which is connected to another mass attached to the other side of a pulley (see figure 2). How heavy can the new mass be so that there is no motion?

This is a sneaky problem. There are now three forces acting on the block–the tension of the rope transmitting the hanging mass's force, the static friction, and gravity. Since the hanging mass is trying to pull the block up the ramp, static friction and gravity are opposing this motion. The effect of the pully is to efficiently transfer the full gravitional force from the hanging mass to the block (think about Newton's third law). Now, it is simple to balance forces,

$$F_s + F_g + F_T = 0 \Rightarrow g\left[-m_1(\mu_s \cos\theta + \sin\theta) + m_2\right] = 0. \tag{6}$$

Plugging in numbers gives a hanging mass of 11.38 kg.

We have now looked at gravity, pulleys, and ramps–are we experts? Not quite. Suppose we put some interesting textures on the surface of the ramp so that the kinetic friction coefficient is not constant. Then we must use calculus. As a simple example, consider instead

$$\mu_k(x) = .2(1 - \frac{x}{L}) \tag{7}$$

where L is the length of the ramp. The origin is the bottom of the ramp which maximizes the kinetic friction. Now, we have the net force as a function of position

$$F_x(x) = F_g[\mu_k(x)\sin\theta - \cos\theta].$$

Rewriting this in terms of $x(t)$ gives

$$\frac{d^2x}{dt^2} = g[(.2\sin\theta)(1-x) - \cos\theta] \tag{8}$$

This is an inhomogeneous linear differential equation. To simplify things, write

$$\frac{d^2x}{dt^2} = -ax + b$$

The solutions to this equation are

$$x(t) = A\cos(\sqrt{a}t) + B\sin(\sqrt{a}t) + \frac{b}{a}.$$

In our case,

$$a = .2g\sin\theta \quad \text{and} \quad b = g(.2\sin\theta - \cos\theta).$$

The initial condition is the block at the top of the ramp at rest. This tells us $A = L - b/a$ and $B = 0$. Thus we have

$$x(t) = \left(1 - \frac{.2\sin\theta - \cos\theta}{.2\sin\theta}\right)\cos(\sqrt{.2g\sin\theta}t) + \frac{.2\sin\theta - \cos\theta}{.2\sin\theta}.$$

Aside: relativity Last time we learned about the effects of spacetime boosts on inertial observations. One peculiar effect of the boost is the relativity of simultaneity. Consider two events, separated in space but not in time, $\vec{x}_1 = (t_1, x_1, 0, 0), \vec{x}_2 = (t_1, x_2, 0, 0)$. The invariant length of these two events is

$$s_1 = \sqrt{x_1^2 - c^2t_1^2}, s_2 = \sqrt{x_2^2 - c^2t_1^2}. \tag{9}$$

After a boost, the events occur at

$$t_1' = s_1\sinh\eta, t_2' = s_2\sinh\eta. \tag{10}$$

Because $s_1 \neq s_2$, $t_1' \neq t_2'$. Since simultaneous events don't remain simultaneous, boosts mix space and time. We would like to understand how the separation in space and in time are related in the two frames. Write

$$\begin{pmatrix} \Delta x \\ \Delta t \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \Delta x' \\ \Delta t' \end{pmatrix}. \tag{11}$$

The motivation for this ansatz is that a boost should be a linear transformation on the space-time position vectors. Linear transformations between vector spaces are given by matrices. Now, to find the matrix, we can take limits where we look at just time dilations, just length contractions, or differences in simultaneity. These lead to

$$A = D = \gamma, \quad B = \gamma v, \quad C = \gamma v/c^2. \tag{12}$$

To summarize, we have the *Lorentz transformations*

$$\Delta x = \gamma(\Delta x' + v\Delta t') \tag{13}$$

$$\Delta t = \gamma(\Delta t' + v\Delta x'/c^2) \tag{14}$$

$$\Delta y = \Delta y' \tag{15}$$

$$\Delta z = \Delta z'. \tag{16}$$

Note that the determinant of the matrix defining the transformation is

$$AD - BC = \gamma^2(1 - v^2/c^2) = 1. \tag{17}$$

Furthermore,

$$-c^2\Delta t^2 + \Delta x^2 = \gamma^2(-c^2\Delta t'^2 - v^2\Delta x'^2/c^2 + \Delta x'^2 + v^2\Delta t'^2 + v\Delta t'\Delta x' - v\Delta t'\Delta x') \tag{18}$$

$$= -c^2\Delta t'^2 + \Delta x'^2. \tag{19}$$

Now, let's ask a question. Suppose an object moves with some speed v_1 relative to a frame S' and the frame S' moves at a speed v_2 relative to a frame S. What is the speed of the object in frame S? The speed is easiest answered by considering two events along the path. Since everything moves at constant velocity, we can calculate the object's speed as

$$u = \frac{\Delta x}{\Delta t}. \tag{20}$$

We also know that in the frame S',

$$v_1 = \frac{\Delta x'}{\Delta t'} \tag{21}$$

Now use the Lorentz transformations,

$$u = \frac{\gamma(\Delta x' + v_2\Delta t')}{\gamma(\Delta t' + v_2\Delta x'/c^2)} = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}. \tag{22}$$

This is the *velocity-addition formula*.

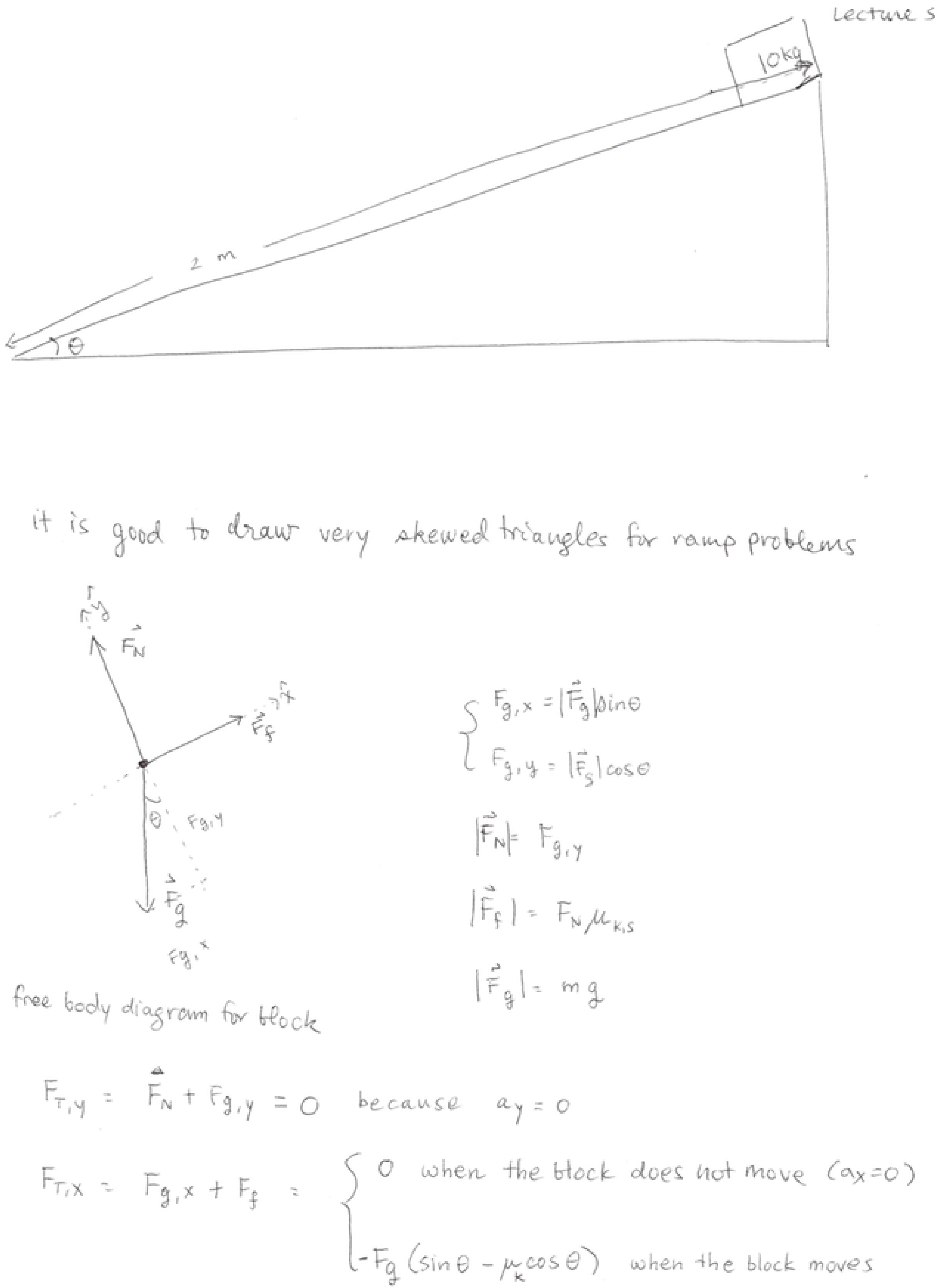


Figure 1: A block on a ramp with friction.

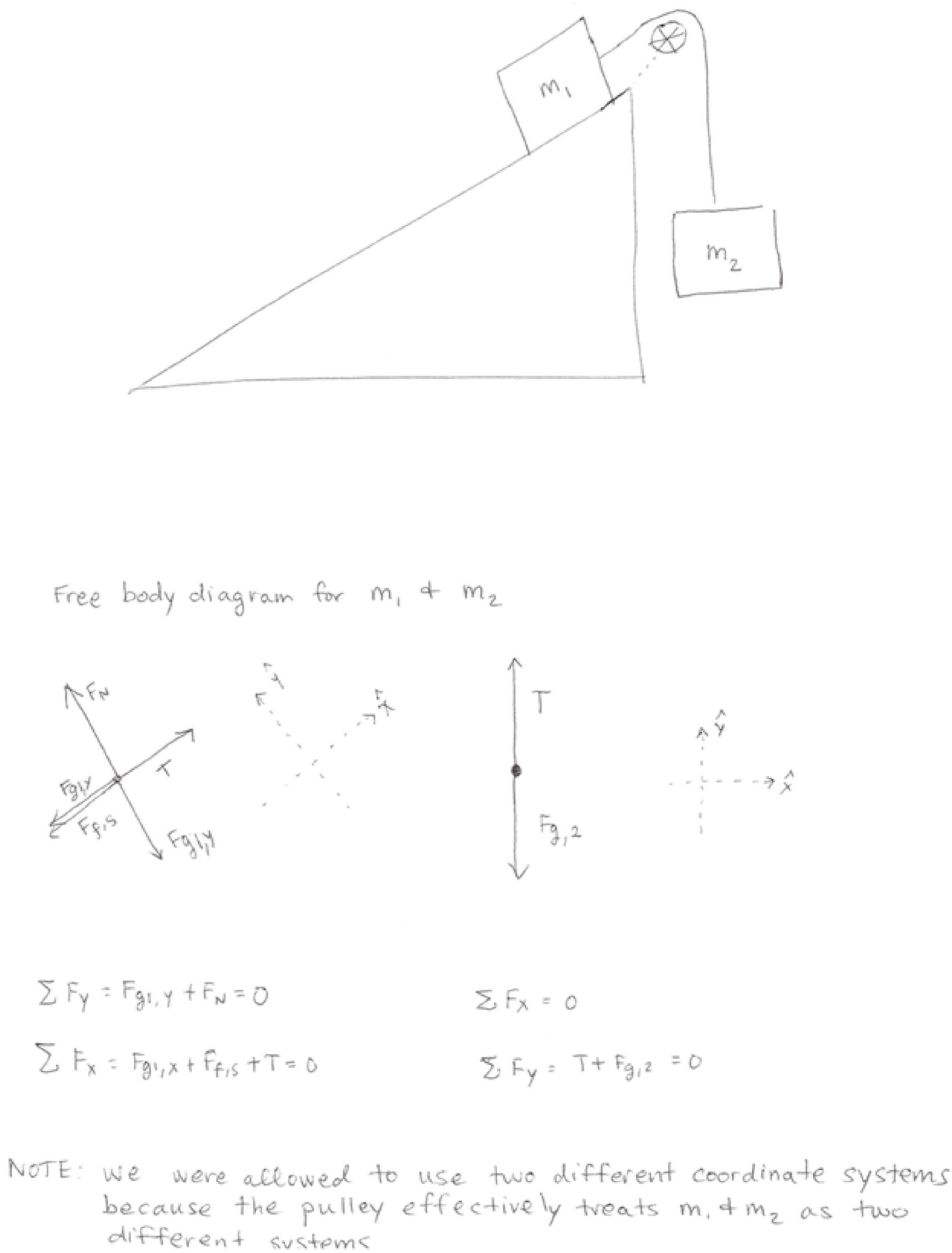


Figure 2: A block on a ramp with friction and a pulley.