Circular motion

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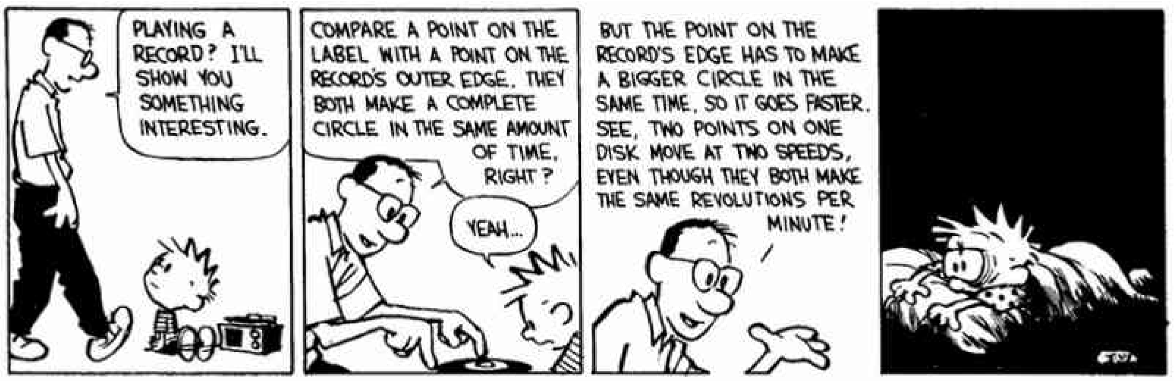

Until now, we have been observers to Newtonian physics through *inertial reference frames*. From our discussion of Newton's laws, these are frames which obey Newton's first law–they don't accelerate and therefore move at constant velocity. Here the rules of vector analysis apply and we can change reference frames so that the frame is not moving. In the context of momentum conservation, we saw that particularly nice inertial frames are those in which one object isn't moving and the frame in which the center of mass is not moving. However, we regularly encounter situations where inertial frames don't apply–for instance when we accelerate. Recall that the acceleration is the change in velocity with respect to time,

$$\vec{a} = \frac{d\vec{v}}{dt}. \tag{1}$$

The change in velocity can be a change in magnitude, as we saw in one dimensional motion, but it also may be a change in direction with no change in magnitude. This occurs in circular motion. Let's see how this works!

A circular worldline for a particle can be written

$$\vec{x}(t) = R(\cos(2\pi t/T), \sin(2\pi t/T), 0). \tag{2}$$

Every $\Delta t = T$, the particle returns to its starting point. The velocity of this worldline is

$$\vec{v}(t) = \frac{2\pi R}{T}(-\sin(2\pi t/T), \cos(2\pi t/T), 0), \tag{3}$$

and its acceleration is

$$\vec{a}(t) = \frac{4\pi^2 R}{T^2}(-\cos(2\pi t/T), -\sin(2\pi t/T), 0) = -\frac{4\pi^2}{T^2}\vec{x}(t). \tag{4}$$

Notice that the last equation is the same differential equation as we saw in the static friction case and the spring–hmm, I wonder if there is a differential equation that describes *periodic motion*. Now, let's calculate the norm of each of these quantities,

$$x(t) = |\vec{x}(t)| = \sqrt{R^2(\cos^2(2\pi t/T) + \sin^2(2\pi t/T))} = R \tag{5}$$

$$v(t) = |\vec{v}(t)| = \sqrt{(\frac{2\pi R}{T})^2(\sin^2(2\pi t/T) + \cos^2(2\pi t/T))} = \frac{2\pi R}{T} = \omega R \tag{6}$$

$$a(t) = |\vec{a}(t)| = \sqrt{(\frac{4\pi^2 R}{T^2})^2(\cos^2(2\pi t/T) + \sin^2(2\pi t/T))} = \frac{4\pi^2 R}{T^2} = \omega^2 R. \tag{7}$$

Here we defined the *angular velocity* $\omega = 2\pi/T$ as the rate at which the particle changes by 1 radian. Let's interpret the norms. The first says that the the norm of particle's position vector is fixed to be the radius of the circle R. This makes sense as the particle is fixed to a circular trajectory. The norm of the velocity, or speed, is a constant which is proportional to the angular velocity. Here we see that it is the direction of the velocity which changes, but not the norm. Finally, the angular acceleration is proportional to the angular velocity squared multiplied by the radius and is constant. How can we have constant acceleration but no change in speed? Let's check the dot products

$$\vec{a}(t) \cdot \vec{v}(t) = \omega^3 R^2 \left[\cos(2\pi t/T)\sin(2\pi t/T) - \cos(2\pi t/T)\sin(2\pi t/T)\right] = 0. \tag{8}$$

While the norm of the acceleration is constant, it always acts exactly perpendicular to direction of motion. Its only impact is to change the direction of the particle. Had it acted in the direction tangential to the worldline, the speed would change. Let's also calculate

$$\vec{v}(t) \cdot \vec{x}(t) = \omega R^2 \left[-\cos(2\pi t/T)\sin(2\pi t/T) + \cos(2\pi t/T)\sin(2\pi t/T)\right] = 0 \tag{9}$$

$$\vec{a}(t) \cdot \vec{x}(t) = -\omega^2 R^2 \left[\cos^2(2\pi t/T) + \sin^2(2\pi t/T)\right] = -\omega^2 R^2. \tag{10}$$

The first of these equations tell us that the velocity is also perpendicular to the particle's position. Since this is motion in a plane, a two-dimensional surface, there are only two basis vectors. If two vectors are perpendicular to the same vector they must be proportional. Thus the acceleration and position vectors are aligned. The last equation tells us that the acceleration vector points in the opposite direction to the position and is larger by a factor of $\omega^2 R$.

The constant norms of these vectors and the fact that the velocity is always perpendicular to the position and acceleration are hints that circular motion is simpler than our description. Let's define two new coordinates, r and θ,

$$x = r\cos\theta, y = r\sin\theta \Rightarrow r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}(y/x). \tag{11}$$

In terms of these coordinates, the circular worldline is

$$r(t) = \sqrt{x(t)^2 + y(t)^2} = R \tag{12}$$

$$\theta(t) = \tan^{-1}\left[\tan(2\pi t/T)\right] = \frac{2\pi t}{T}. \tag{13}$$

In these coordinates, circular motion is one dimensional–the radial coordinate does not change. The angular coordinate changes linearly with time, similar to constant velocity motion in rectilinear coordinates. In fact, the angular velocity

$$\omega = \frac{d\theta}{dt} = \frac{2\pi}{T} \tag{14}$$

is the same as we saw before and is constant! Furthermore,

$$\alpha = \frac{d\omega}{dt} = 0 \tag{15}$$

showing that the particle doesn't speed up as it goes around the circle. For fun, let's investigate what linear motion looks like in these coordinates

$$\vec{x}(t) = (x_0, y_0 + vt, 0). \tag{16}$$

Then,

$$r(t) = \sqrt{x_0^2 + (y_0 + vt)^2} \tag{17}$$

$$\theta(t) = \tan^{-1} \frac{y_0 + vt}{x_0} \tag{18}$$

This is complicated. There is both an angular and radial velocity. The radial velocity is

$$\frac{dr}{dt} = \frac{v(y_0 + vt)}{r(t)} \tag{19}$$

and the angular velocity

$$\frac{d\theta}{dt} = \frac{v}{x_0 \left(\frac{(tv+y_0)^2}{x_0^2} + 1 \right)}. \tag{20}$$

Linear motion is complicated in these coordinates (a good sign that this is *not* and inertial reference frame). For physics problems, a good sanity check is to look at limits. Let's take the $t \to \infty$ limit of the above expressions.

$$\lim_{t\to\infty} r(t) = y_0 + vt + \mathcal{O}(1/t) \tag{21}$$

$$\lim_{t\to\infty} \theta(t) = \pi/2 + \mathcal{O}(\frac{1}{t^2}) \tag{22}$$

$$\lim_{t\to\infty} \frac{dr}{dt} = v + \mathcal{O}(\frac{1}{t^2}) \tag{23}$$

$$\lim_{t\to\infty} \frac{d\theta}{dt} = 0 + \mathcal{O}(\frac{1}{t^2}). \tag{24}$$

In other words, after a long time, no matter the inital x_0, the particle is confined very close to the y-axis in circular coordinates and its angle changes slower and slower with time. It moves along the axis at constant speed where r and y serve as basically the same coordinate.

How did I arrive at these limits? I used a Taylor expansion. Recall from your calculus class that a function can be defined completely in terms of its derivatives at a point. We want to express any function as a polynomial (of which is easy to take derivatives) whose derivatives exactly match the function at a point. Near this point, the polynomial expansion should be a good approximation to the real function and is often times exact. Consider a function $f(x)$. Near $x = x_0$, we write

$$f(x) \approx f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0) + \frac{1}{2} \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} (x - x_0)^2 + ... \tag{25}$$

$$= f(x_0) + \sum_{n=1}^{\infty} \left.\frac{d^n f}{dx^n}\right|_{x=x_0} \frac{(x - x_0)^n}{n!}. \tag{26}$$

Let's check that the derivatives match. At $x = x_0$, any factor of $(x - x_0)^m$ with $m > 0$ will vanish. Taking l derivatives of $(x - x_0)^m$ with $l > m$ also vanishes. Thus, we can look at the l^{th} derivative of this expression

$$\frac{d^l f}{dx^l} = \sum_{n=1}^{\infty} \frac{d^n f}{dx^n}\bigg|_{x=x_0} \frac{(x - x_0)^{n-l} l!}{n!}, \tag{27}$$

and plugging in $x = x_0$,

$$\frac{d^l f}{dx^l}\bigg|_{x=x_0} = \sum_{n=1}^{\infty} \frac{d^n f}{dx^n}\bigg|_{x=x_0} \delta_{nl} = \frac{d^l f}{dx^l}\bigg|_{x=x_0} \checkmark. \tag{28}$$

So the expansion works. For the expansion above, we expanded in t around $t = \infty$ and kept only the terms that did not vanish in this limit. The notation $\mathcal{O}(t^n)$ says that the terms we do not write down are a series in t that start at t^n and are smaller than the terms that we did write down.

Circular motion obeys the same equations as linear motion. For instance, for a particle moving at constant angular acceleration, we can find its angular position by replacing $x \to \theta, v \to \omega, a \to \alpha$,

$$\theta(t) - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2. \tag{29}$$

Obviously, a particle moving in a purely radial direction is the same as linear motion, because we can always rotate our coordinate system so that the rectilinear coordinates align with the direction of motion.

Earlier we argued that symmetries of space(time) lead to conservation laws–for instance conservation of energy and conservation of momentum. Along a circular path, we are free to define $\theta = 0$ anywhere along the path. This is a *rotational symmetry*. Such a symmetry also leads to a conservation law–conservation of *angular momentum*. Like linear momentum, angular momentum is a vector. It is defined as

$$\vec{L} = \vec{r} \times \vec{p}. \tag{30}$$

Let's find a simple angular momentum to gain some insight. Consider a particle of mass m moving in the circular trajectory of eq. 2. The particles momentum is

$$\vec{p} = m\vec{v}(t) = m\frac{2\pi R}{T}\left[-\sin(2\pi t/T)\hat{x} + \cos(2\pi t/T)\hat{y}\right]. \tag{31}$$

The angular momentum is

$$\vec{L} = m\frac{2\pi R^2}{T}\Big[-\sin(2\pi t/T)\cos(2\pi t/T)(\hat{x} \times \hat{x}) + \sin(2\pi t/T)\cos(2\pi t/T)(\hat{y} \times \hat{y}) \tag{32}$$

$$+ \cos(2\pi t/T)\cos(2\pi t/T)(\hat{x} \times \hat{y}) - \sin(2\pi t/T)\sin(2\pi t/T)(\hat{y} \times \hat{x})\Big] \tag{33}$$

$$= m\frac{2\pi R^2}{T}\left[\cos^2(2\pi t/T)\hat{z} + \sin^2(2\pi t/T)\hat{z}\right] = m\frac{2\pi R^2}{T}\hat{z} \tag{34}$$

$$= m\omega R^2 \hat{z} \tag{35}$$

To arrive at this result, we had to realize $\hat{x_i} \times \hat{x_i} = 0$ and $\hat{x} \times \hat{y} = -\hat{y} \times \hat{x} = \hat{z}$. An easy way to remember these results is the "right-hand rule" from earlier. Notice also that the angular momentum,

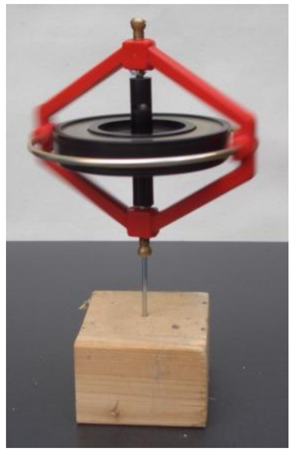

Figure 1: A gyroscope stays standing because it tries to conserve angular momentum along the vertical axis. If it fell over, angular momentum would move to a different axis and not be conserved. Eventually it falls because friction slows down the wheel's rotation (image credit: wikipedia)

defined through a cross product, obeys the right hand rule–point your fingers in the radial direction and sweep them in the direction of motion (counterclockwise). The direction your thumb points is the direction of angular momentum. We can simplify this expression by defining the angular momentum vector, which is a vector that has magnitude ω and points along the axis of rotation. In this case,

$$\vec{\omega} = \omega \hat{z} \tag{36}$$

and

$$\vec{L} = mR^2 \vec{\omega}. \tag{37}$$

It may seem surprising to you that the angular momentum is not in the plane defined by the radial vector and the velocity. In fact it is always perpendicular to to this direction. Consider, however, a gyroscope (figure 1). A spinning wheel can remain standing on its rotational axis because the universe tries to conserve angular momentum along this direction. For completeness, let's also find the angular momentum in circular coordinates. We will actually do this in a roundabout way to derive the basis decomposition of a vector in circular coordinates. Because angular momentum is aligned perpendicular to the plane of the circle, it can't depend on our choice of coordinates.

$$\vec{x}(t) = R\hat{r} \quad \text{and} \quad \vec{p} = \left[p_r \hat{r} + p_\phi \hat{\phi} \right]. \tag{38}$$

We have two unknowns–ω_r and ω_ϕ. To find these, we need two equations. Fortunately, we have two,

$$\vec{L} = m\omega R^2 \hat{z} \quad \text{and} \quad |\vec{p}| = m\omega R. \tag{39}$$

Now,

$$|\vec{p}|^2 = m \left[p_r^2 \hat{r} \cdot \hat{r} + p_\phi^2 \hat{\phi} \cdot \hat{\phi} \right] \tag{40}$$

and

$$\vec{L} = Rp_\phi(\hat{r} \times \hat{\phi}) = Rp_\phi \hat{z}, \tag{41}$$

where we again used the right hand rule ($\hat{\phi}$ always points counterclockwise). Then

$$p_\phi = m\omega R. \tag{42}$$

Clearly, to satisfy $|\vec{p}| = m\omega R$, $p_r = 0$. Along the way, we learned the transformation law between the cartesian and circular basis vectors. We did? Recall that vectors are exist independent of coordinate choices (though their components along basis vectors may look very different). For instance

$$\vec{p} = m\omega R\left[-\sin(2\pi t/T)\hat{x} + \cos(2\pi t/T)\hat{y}\right] = m\omega R\hat{\phi}. \tag{43}$$

To find the relationship, we notice a bizarre feature that the $x - y$ coordinates have time dependent components and the $r - \phi$ coordinates do not. This is because there is a time dependence hiding in $\hat{\phi}$. In switching between coordinate systems, there is a simple transformation law,

$$\hat{x} = \frac{\partial x}{\partial \phi}\hat{\phi} + \frac{\partial x}{\partial r}\hat{r} \tag{44}$$

$$\hat{y} = \frac{\partial y}{\partial \phi}\hat{\phi} + \frac{\partial y}{\partial r}\hat{r}. \tag{45}$$

Notice that this can be written in a simple way as

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial r} \end{pmatrix} \begin{pmatrix} \hat{\phi} \\ \hat{r} \end{pmatrix} \tag{46}$$

The matrix in this transformation is called the "Jacobian" of the transformation. Let's check that this works for our rotating particle.

$$\frac{\partial x}{\partial \phi(t)} = -\sin(\phi(t)), \quad \frac{\partial y}{\partial \phi(t)} = \cos(\phi(t)) \tag{47}$$

$$\frac{\partial x}{\partial r} = \cos(\phi(t)), \quad \frac{\partial y}{\partial r} = \sin(\phi(t)). \tag{48}$$

Then

$$\vec{p} = m\omega R\hat{\phi} = m\omega R[-\sin(2\pi t/T)\hat{x} + \cos(2\pi t/T)\hat{y}]\checkmark. \tag{49}$$

We can also invert the Jacobian to go from $\hat{x}, \hat{y}$ to $\hat{\phi}, \hat{r}$. Try this for fun.

Newton's laws also apply to circular motion but in a slightly surprising way. Pretending for a moment that we are Isaac Newton (he was only 24 when he invented calculus, so about our ages) let's ask about the rate of change of the angular momentum. Recall that this was how he defined linear force.

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}. \tag{50}$$

The first term in this equation vanishes. This is because $\vec{p} = md\vec{r}/dt$ and the cross product of a vector with itself. This *angular force* is called *torque*. For our circularly moving particle, the torque vanishes. This is most easily seen in circular coordinates where it is obvious that

$$\frac{d\vec{r}}{dt} = 0. \tag{51}$$

In cartesian coordinates,

$$\frac{d\vec{p}}{dt} = -m\omega^2 R\left[\cos(2\pi t/T)\hat{x} + \sin(2\pi t/T)\hat{y}\right]. \tag{52}$$

Taking the cross product of this with $\hat{r}$ gives

$$\vec{\tau} = -m\omega^2 R^2\left[\cos(2\pi t/T)\sin(2\pi t/T)\hat{z} - \cos(2\pi t/T)\sin(2\pi t/T)\hat{z}\right] = 0\checkmark \tag{53}$$

Let's look at an example which does not have vanishing torque. Consider

$$\vec{r}(t) = (R, \frac{R}{2}\alpha t^2, 0). \tag{54}$$

Now,

$$\vec{p}(t) = mR(0, \alpha t, 0) \quad \text{and} \quad \frac{dp}{dt} = mR\alpha\hat{\theta} \tag{55}$$

so that

$$\vec{\tau} = mR^2\alpha\hat{z} = mR^2\vec{\alpha} \tag{56}$$

This looks like $\vec{F} = m\vec{a}$ if we define that angular mass, called the *moment of inertia*, $I = mR^2$. Note that, like the angular momentum, the torque is perpendicular to the plane defined by the position and velocity vectors, and we defined the angular acceleration vector $\vec{\alpha}$, as the vector with norm α pointing along this axis. In general, when we have a complicated mass distribution, $m(\vec{r})$, spinning about some axis, the moment of inertia is

$$I = \int r^2 dm \tag{57}$$

As an example, let's consider a disk of radius R and mass M. It's mass density is constant, just the total mass divided by the total area:

$$m(r) = \frac{M}{\pi R^2} \tag{58}$$

Its moment of inertia is

$$I = \frac{M}{\pi R^2}\int_{\text{disc}} r^2 dA = \frac{M}{\pi R^2}\int_0^{2\pi} d\theta \int_0^R r^3 = \frac{MR^2}{2}. \tag{59}$$

It is easier to spin a disc of mass M and radius R than it is to rotate a particle of the same mass at this radius. This is because the mass is distributed about the rotational axis. If you have ever watched the shotput event in the olympics, where do the athletes hold the weight? Why?

Written in terms of the moment of inertia, the angular momentum is

$$\vec{L} = I\vec{\omega}. \tag{60}$$

As before, the cross-product makes things a little messier, but otherwise this looks just like the linear momentum.

Important distinctions The *centripetal force* is the force required to maintain an object in constant circular motion. Its magnitude is

$$F_c = \frac{mv^2}{r} = m\omega^2 r. \tag{61}$$

The *centrifugal force* is an imaginary force which does not exist. Because of the equivalence principle, we are used to thinking in terms of inertial frames. Constant angular velocity is not an inertial frame. It is accelerating and because Newton's first law makes us want to move in a straight line, in the non-inertial frame this feels like a force. It is not.