Tight Noise Thresholds for Quantum Computation with Perfect Stabilizer Operations

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Work with **Wim van Dam** [WvD,MH: Phys. Rev. Lett. **103** 170504 (2009), arXiv:0907.3189]

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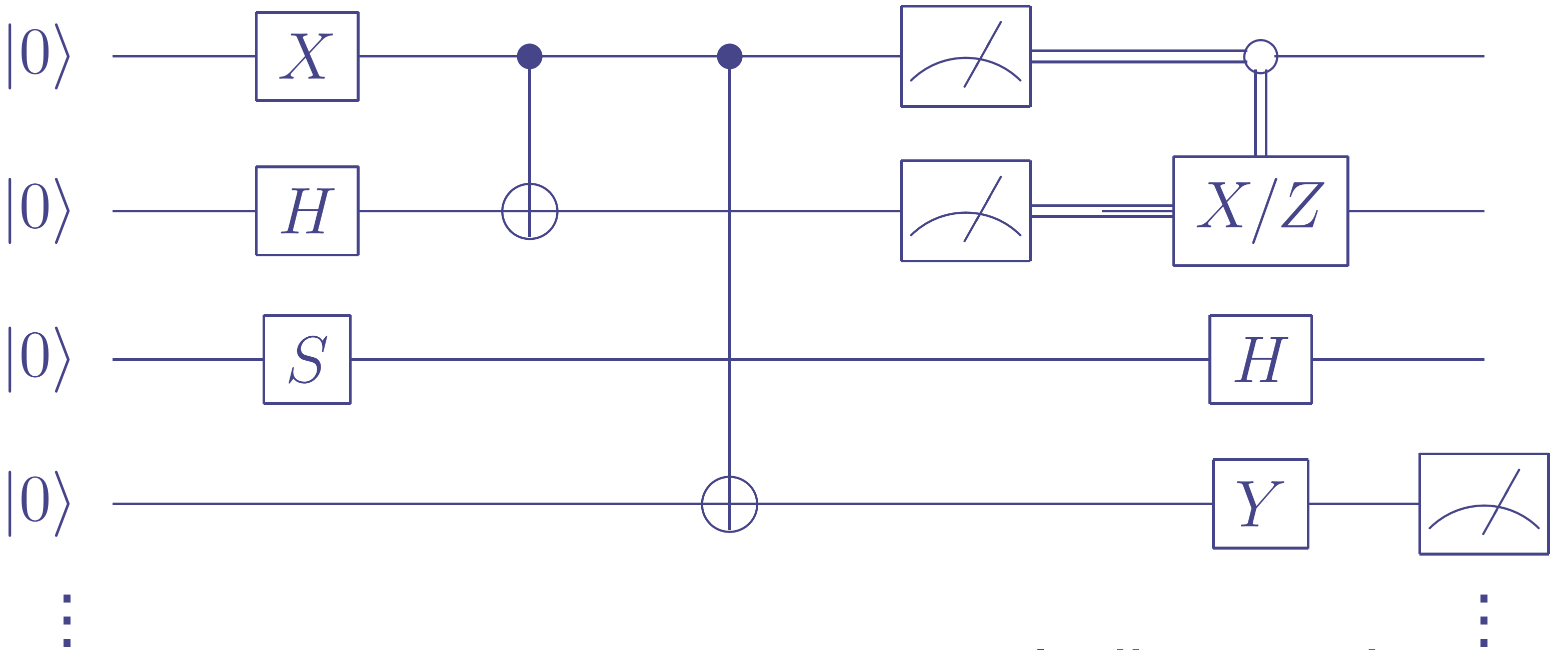
Perfect Stabilizers?

For this talk: stabilizer operations are perfect.

- Low enough error rate + using a suitable error correcting code $\Rightarrow$ effectively perfect.
- Sometimes physical set-up gives you $\approx$ perfect stabilizers:
  - $*$ Pfaffian ($\nu = \frac{5}{2}$) fractional Quantum Hall system
  - $*$ "Protected" superconducting qubit
- See if tight threshold even possible in principle

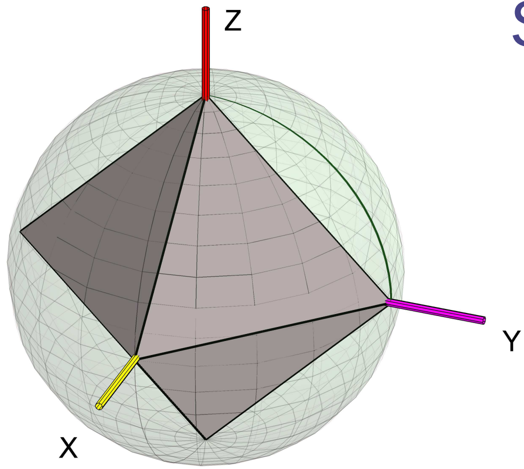
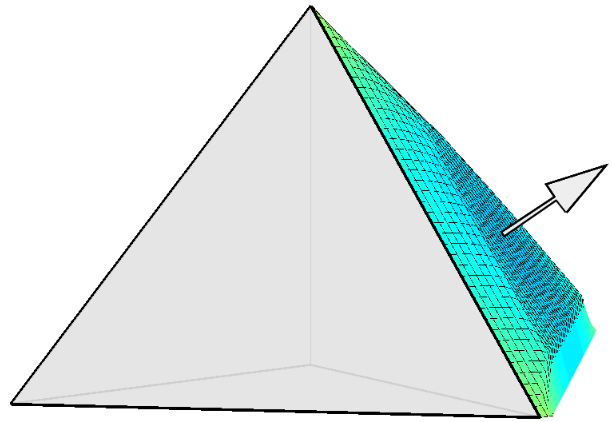
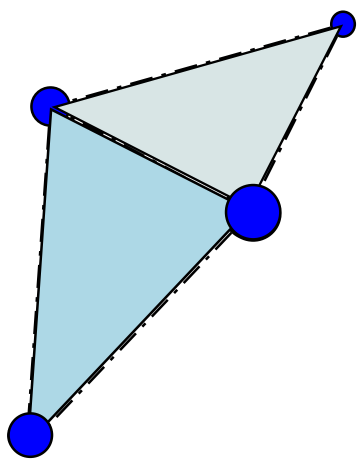
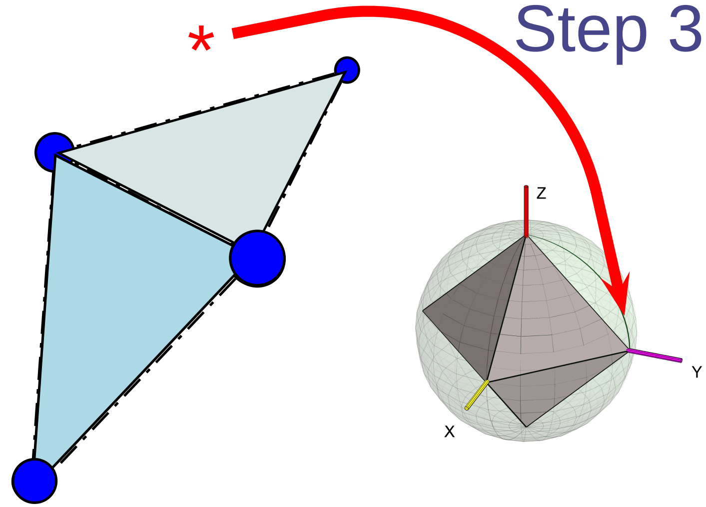
Gottesman Knill Theorem



- n Input Qubits
- Simulate in $\mathcal{O}(n^2)$ time on classical computer

Including the 24 allowed single-qubit (**Clifford**) gates $= \{S, H, S^2, SHS, \ldots\}$

Noise Bounds for States and Operations

<table>
<tr><th></th><th>Upper Bound:</th><th>Lower Bound:</th></tr>
<tr><td>States:
Octahedron</td><td>Step 0

</td><td>Step 1
</td></tr>
<tr><td>Operations:
Clifford Polytope</td><td>Step 2
</td><td>Step 3

</td></tr>
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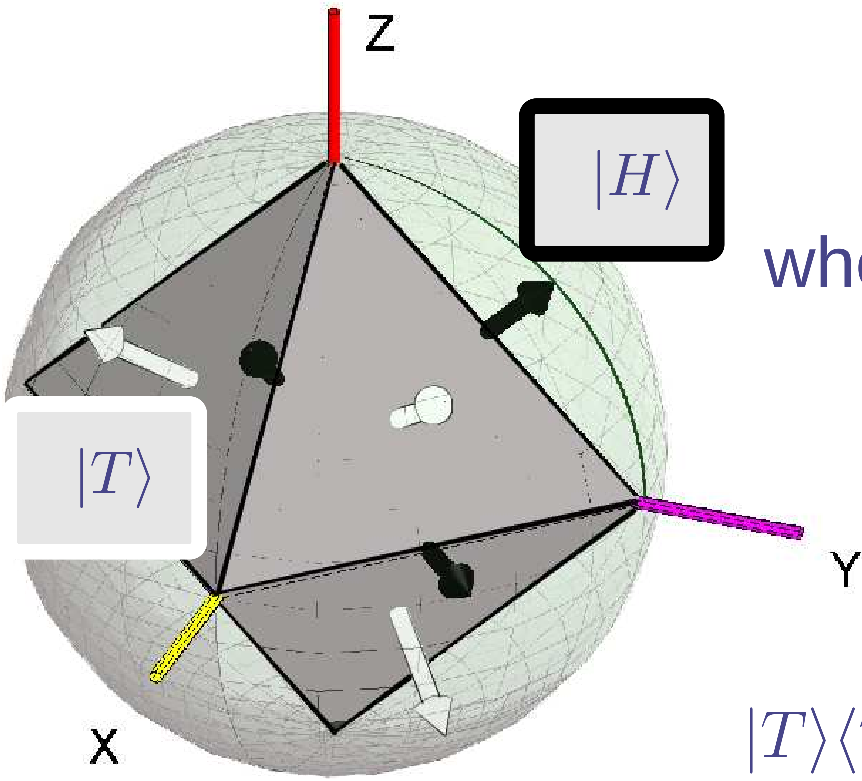
Step 1: Noisy Ancillae can enable UQC



$$|H\rangle\langle H| = \frac{1}{2}\left(I + \frac{1}{\sqrt{2}}(\pm\sigma_i \pm \sigma_j)\right)$$

where $\sigma_i, \sigma_j \in \{\sigma_x, \sigma_y, \sigma_z\}, \sigma_i \neq \sigma_j$

Magic States

$$|T\rangle\langle T| = \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(\pm\sigma_x \pm \sigma_y \pm \sigma_z)\right)$$

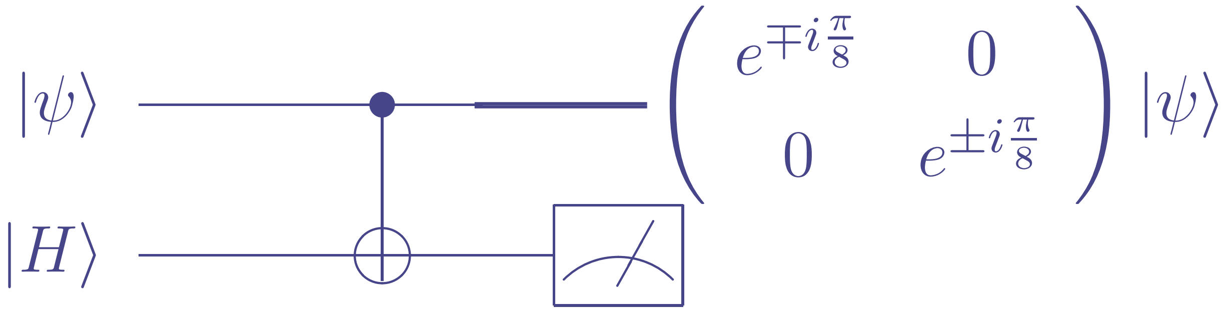
Or, for example, in ket form:

$|H\rangle = |0\rangle + e^{\frac{i\pi}{4}}|1\rangle$

$|T\rangle = \cos(\vartheta)|0\rangle + e^{\frac{i\pi}{4}}\sin(\vartheta)|1\rangle$ with $\cos(2\vartheta) = \frac{1}{\sqrt{3}}$

Step 1: Noisy Ancillae can enable UQC

- Pure magic states + perfect stabilizer operations enable UQC.
- Impure magic states can be distilled towards pure magic states using stabilizer operations only.
- Access to $|H\rangle$ states enables performing "$\pi/8$" gate:



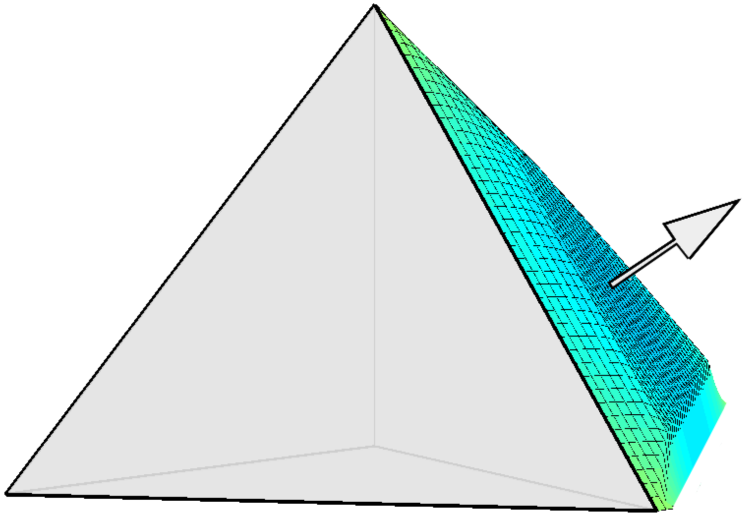
(Using $|T\rangle$ states enables a different gate)

Bravyi, S. and A. Kitaev, "Universal quantum computation with ideal Clifford gates and noisy ancillas" Phys. Rev. A **71**, 022316 2005

Step 1: Noisy Ancillae can enable UQC

- States with Bloch vectors satisfying $\max\{|x|+|z|, |x|+|y|, |y|+|z|\} > 1$ are distillable (tight up to the 12 edges of octahedron).

[Rei1] Ben W. Reichardt, "Improved magic states distillation for quantum universality", Quantum Information Processing 4 pp.251-264 (2005).



- There is currently an undistillable region just outside the octahedron faces (Bloch vector: $1 < |x|+|y|+|z| < \frac{3}{\sqrt{7}}$).

[CB'09] Earl T. Campbell and Dan E. Browne, "Bound states for magic state distillation", arXiv:0908.0836 (2009)

Step 2: Clifford Polytope

General Idea

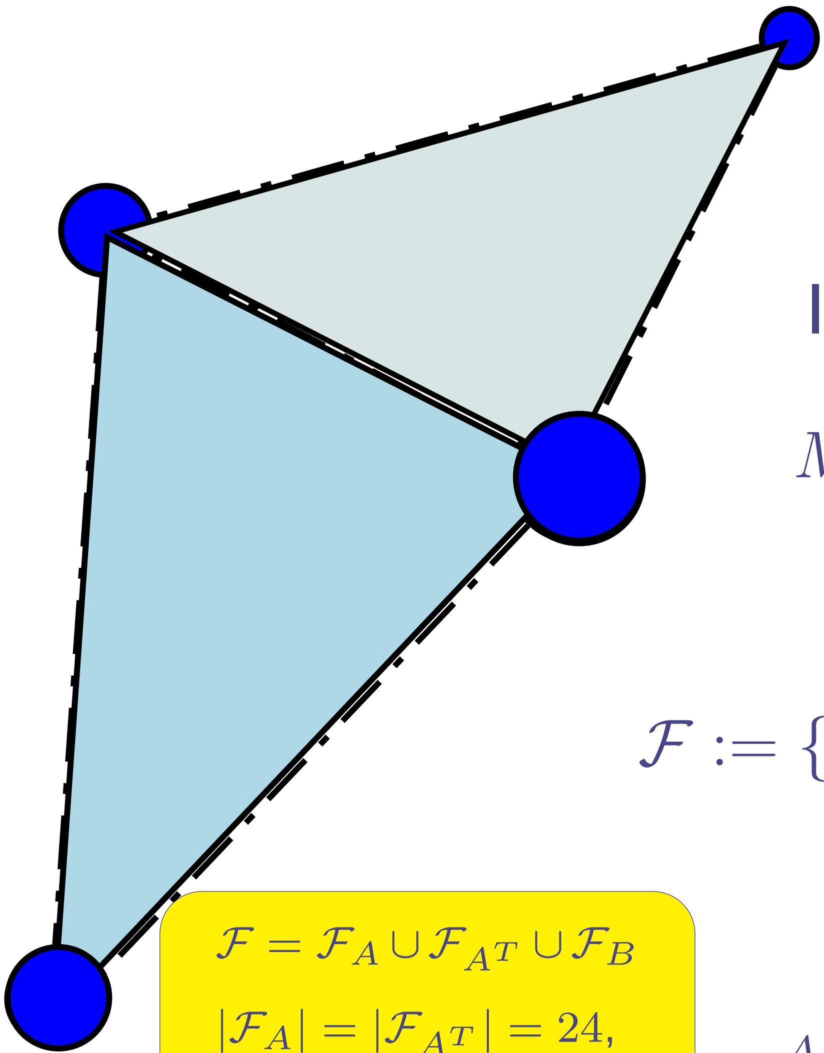
- Recall U outside Clifford group enables UQC.
- Noise during implementation of U means

$$
\begin{aligned}
SU(2)\ \text{picture}: &\quad \mathcal{E}_{\text{TOTAL}}(\rho) &= (1-p)U\rho U^\dagger + p\mathcal{E}_{\text{NOISE}}(\rho) \\
SO(3)\ \text{picture}: &\quad M &= (1-p)R + pN
\end{aligned}
$$

- For what noise rate, p, is $\mathcal{E}_{\text{TOTAL}}(\rho)$ implementable using Clifford operations only?
- Depends on U. Depends on $\mathcal{E}_{\text{NOISE}}(\rho)$.

H. Buhrman, R. Cleve, M. Laurent, N. Linden, A. Schrijver and F. Unger, "New limits on fault-tolerant quantum computation", FOCS 411(2006).

Step 2: Clifford Polytope



M is an $\mathbb{R}^{3\times 3}$ transformation.

Is $M = \sum_{i=1}^{24} p_i C_i \ \ (\sum_{i=1}^{24} p_i = 1)$?

$M \in$ Polytope $: M \cdot F \leq 1 \quad \forall F \in \mathcal{F}$

where

$$\mathcal{F} := \{C_1 F C_2 | C_1, C_2 \in \mathcal{C}, F \in \{A, A^T, B\}\}$$

with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

$\mathcal{F} = \mathcal{F}_A \cup \mathcal{F}_{A^T} \cup \mathcal{F}_B$

$|\mathcal{F}_A| = |\mathcal{F}_{A^T}| = 24,$

$|\mathcal{F}_B| = 72$

Our Results:

(i) (Under **Unital** Noise:)

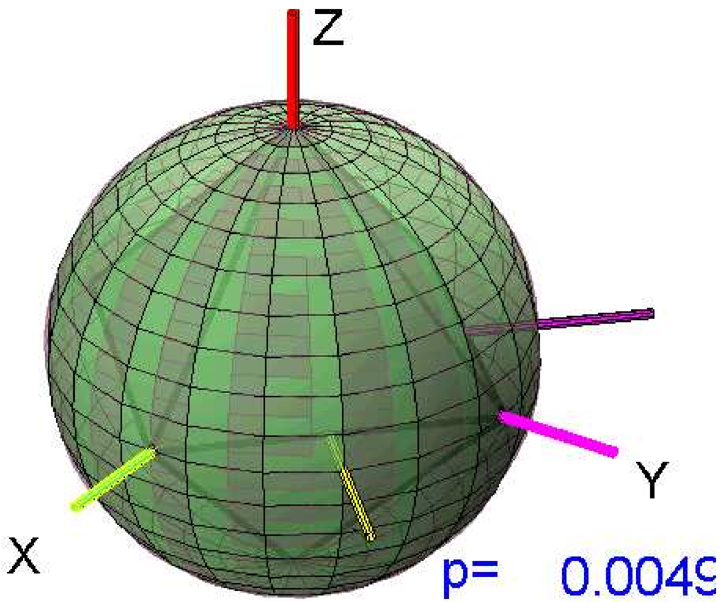
UQC for single qubit gates reduces to UQC for single qubit states

(ii) (Under **Depolarizing** Noise:)

All operations outside the Clifford polytope enable UQC

Step 3: Noisy gates can enable UQC

General Idea: Apply noisy U to an input stabilizer state.



For example:

- Apply noisy "$\pi/8$" gate to σ_x eigenstate. . . Outside octahedron for up to 29% depolarizing noise
- Upper Bound from Clifford polytope [BCLLSU]: $p \geq 0.453 \Rightarrow$ UQC

Can we get closer to 45%?

[Rei2] Ben W. Reichardt, "Quantum universality by state distillation", QIC Vol.9 (2009).

Step 3: Noisy gates can enable UQC

- Apply noisy "$\pi/8$" gate to half of $\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

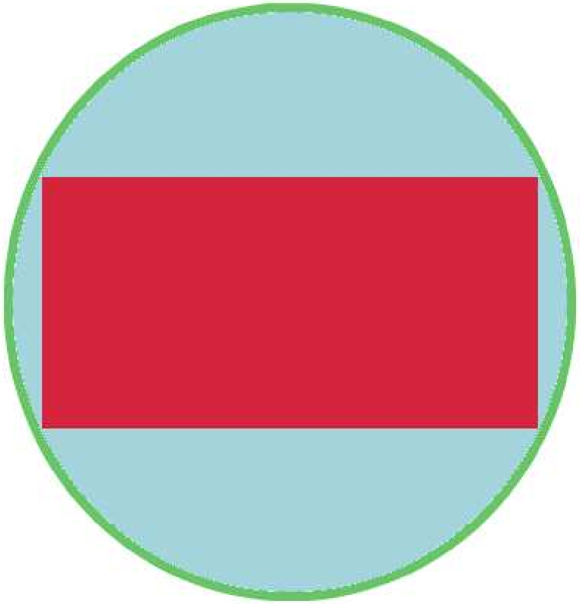
1. Perform parity measurement $\Pi = \frac{1}{2}\left(II + \sigma_z\sigma_z\right)$ on output .
2. Resulting state is single qubit ρ:

$$\rho = \begin{pmatrix} \frac{1}{2} & \frac{(1-p)e^{-i\frac{\pi}{4}}}{(2-p)} \\ \frac{(1-p)e^{i\frac{\pi}{4}}}{(2-p)} & \frac{1}{2} \end{pmatrix}$$

3. Check if the output state ρ has a Bloch vector that enables distillation.

$|\mathbf{x}| + |\mathbf{y}| > \mathbf{1}$ for $\mathbf{p} < \mathbf{0.453}$ . . . a tight bound [Rei2]

Step 3: Noisy gates can enable UQC

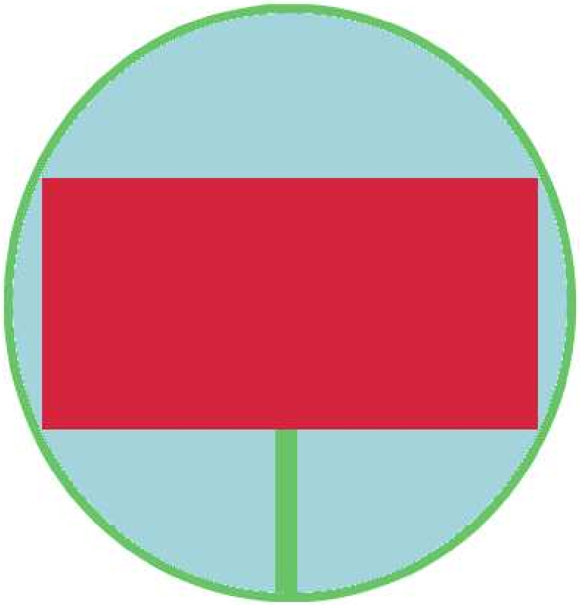


Depolarizing Noise:

$$\mathcal{E}(\rho) = (1-p)U\rho U^\dagger + p\frac{I}{2}$$

[BCLLSU]: $p \geq 0.453 \Rightarrow$ UQC (All U)

Step 3: Noisy gates can enable UQC



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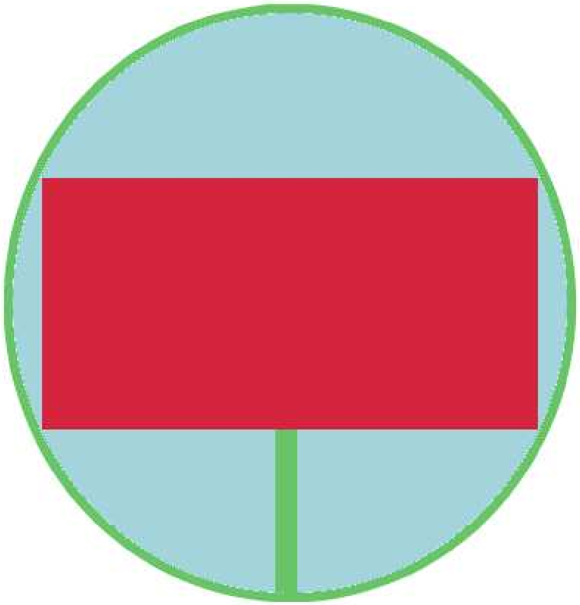
[BCLLSU]: $p \geq 0.453 \Rightarrow$ UQC (All U)

[Rei2]: $p < 0.453 \Rightarrow$ UQC ("$\pi/8$" gate)

"$\pi/8$" gate tight

Step 3: Noisy gates can enable UQC



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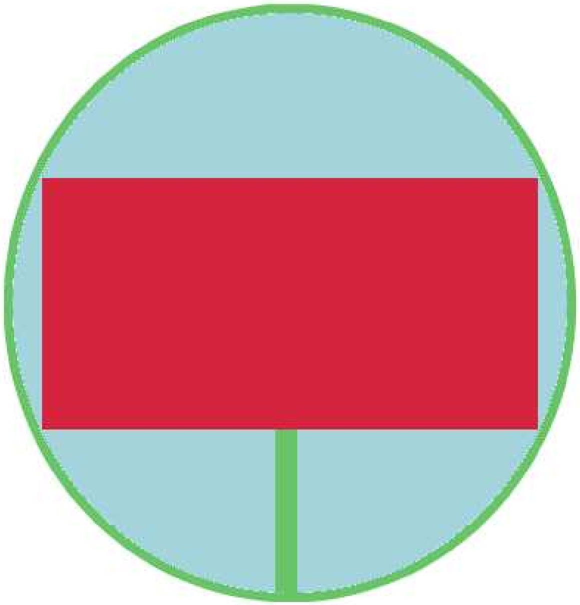
What about other gates?

"$\pi/8$" gate tight

Step 3: Noisy gates can enable UQC



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Depolarizing Noise:

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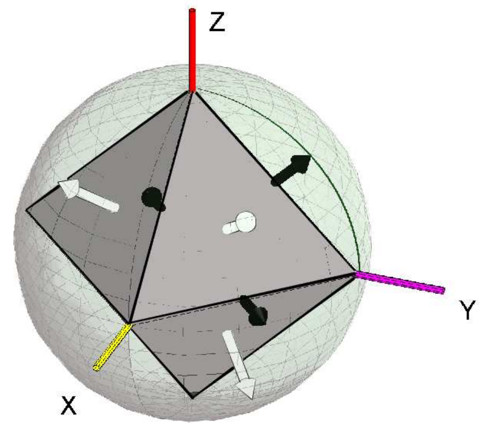
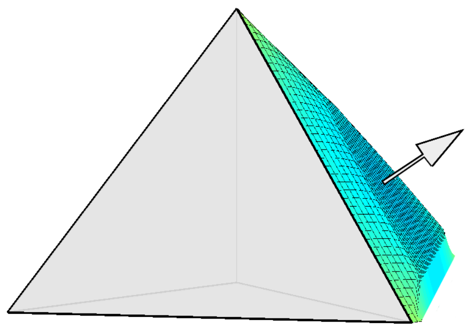
What about other gates?

Similar to ancilla question:





Step 3 Proof: Interpreting Polytope Facets

- Recall:

$$\begin{aligned} SU(2)\ \text{picture}: \quad \mathcal{E}(\rho) &= (1-p)U\rho U^\dagger + p\mathcal{E}_{\text{NOISE}}(\rho) \\ SO(3)\ \text{picture}: \quad M &= (1-p)R + pN \end{aligned}$$

- Define: $\varrho = \mathcal{I} \otimes \mathcal{E}(|\Phi\rangle\langle\Phi|)$

$\left(\text{Jamiolkowski}: |\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)$

- Pauli Decomposition:

$\varrho = \frac{1}{4}\sum_{i,j} \mathsf{c}_{ij}(\sigma_i \otimes \sigma_j) \qquad i,j \in \{I,X,Y,Z\}$

$$\Rightarrow \mathsf{c}_{IX} = \mathsf{c}_{IY} = \mathsf{c}_{IZ} = \mathsf{c}_{XI} = \mathsf{c}_{YI} = \mathsf{c}_{ZI} = 0, \quad \mathsf{c}_{II} \equiv 1$$

$$\Rightarrow M = \begin{pmatrix} \mathsf{c}_{XX} & -\mathsf{c}_{YX} & \mathsf{c}_{ZX} \\ \mathsf{c}_{XY} & -\mathsf{c}_{YY} & \mathsf{c}_{ZY} \\ \mathsf{c}_{XZ} & -\mathsf{c}_{YZ} & \mathsf{c}_{ZZ} \end{pmatrix}$$

Step 3 Proof: Interpreting "A-type" Facets

- Perform Weight 1 Stabilizer Measurement on ϱ
  $\Rightarrow$ Postselect to get single qubit state ρ'

  e.g. Measurement $\Pi = \frac{1}{2}(II + XI)$ returns

  $$\vec{r}(\rho') = \left(\frac{\mathsf{c}_{XX}}{\mathsf{c}_{II}}, \frac{\mathsf{c}_{XY}}{\mathsf{c}_{II}}, \frac{\mathsf{c}_{XZ}}{\mathsf{c}_{II}}\right).$$

- Check if ρ' outside octahedron ($||\vec{r}(\rho')||_1 > 1$?)

  $|\mathsf{c}_{\mathbf{XX}}| + |\mathsf{c}_{\mathbf{XY}}| + |\mathsf{c}_{\mathbf{XZ}}| > \mathsf{c}_{\mathbf{II}}$? $\qquad (\mathsf{c}_{\mathbf{II}} \equiv \mathbf{1})$

$$\begin{pmatrix} \mathsf{c}_{XX} & -\mathsf{c}_{YX} & \mathsf{c}_{ZX} \\ \mathsf{c}_{XY} & -\mathsf{c}_{YY} & \mathsf{c}_{ZY} \\ \mathsf{c}_{XZ} & -\mathsf{c}_{YZ} & \mathsf{c}_{ZZ} \end{pmatrix} \cdot \begin{pmatrix} \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix} > 1\ ?$$

Step 3 Proof: Interpreting "A-type" Facets

- Perform Weight 1 Stabilizer Measurement on ϱ

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- Check if ρ' outside octahedron ($||\vec{r}(\rho')||_1 > 1$?)

  $|c_{\mathbf{XX}}| + |c_{\mathbf{XY}}| + |c_{\mathbf{XZ}}| > c_{\mathbf{II}}$? $\qquad (c_{\mathbf{II}} \equiv \mathbf{1})$

$$\left(M \right) \cdot \left(F \in \mathcal{F}_A \right) > 1?$$

Step 3 Proof: Interpreting "B-type" Facets

- Perform Weight 2 Stabilizer Measurement on ϱ
  $\Rightarrow$ Postselect to get single qubit state ρ'

  e.g. Measurement $\Pi = \frac{1}{2}(II + YX)$ returns

$$\vec{r}(\rho') = \left(0, \frac{c_{XZ} - c_{ZY}}{c_{II} + c_{YX}}, -\frac{c_{XY} + c_{ZZ}}{c_{II} + c_{YX}}\right).$$

- Check if ρ' outside octahedron ($||\vec{r}(\rho')||_1 > 1$?)

  $|c_{\mathbf{XZ}} - c_{\mathbf{ZY}}| + |-(c_{\mathbf{XY}} + c_{\mathbf{ZZ}})| - c_{\mathbf{YX}} > c_{\mathbf{II}}$? $\qquad (c_{\mathbf{II}} \equiv \mathbf{1})$

$$\begin{pmatrix} c_{XX} & -c_{YX} & c_{ZX} \\ c_{XY} & -c_{YY} & c_{ZY} \\ c_{XZ} & -c_{YZ} & c_{ZZ} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} > 1 \text{ ?}$$

Step 3 Proof: Interpreting "B-type" Facets

- Perform Weight 2 Stabilizer Measurement on ϱ

  $\Rightarrow$ Postselect to get single qubit state ρ'

  e.g. Measurement $\Pi = \frac{1}{2}(II + YX)$ returns

$$\vec{r}(\rho') = \left(0, \frac{c_{XZ} - c_{ZY}}{c_{II} + c_{YX}}, -\frac{c_{XY} + c_{ZZ}}{c_{II} + c_{YX}}\right).$$

- Check if ρ' outside octahedron ($||\vec{r}(\rho')||_1 > 1$?)

  $|c_{\mathbf{XZ}} - c_{\mathbf{ZY}}| + |-(c_{\mathbf{XY}} + c_{\mathbf{ZZ}})| - c_{\mathbf{YX}} > c_{\mathbf{II}}$? $\quad (c_{\mathbf{II}} \equiv \mathbf{1})$

$$\left(M \right) \cdot \left(F \in \mathcal{F}_B \right) > 1 \text{ ?}$$

Step 3 Proof: Unital Case Completed

I. Tight Qubit Ancilla Threshold $\Longrightarrow$ Tight Noise Threshold (All Unital Operations)

- Depending on which facet $M = (1-p)R + pN$ violates:

$[\mathcal{F}_A]$: $\mathcal{E}$ applied to half $|\Phi\rangle$ then measure weight 1 stab op. $\rightarrow$ Outside Face

$[\mathcal{F}_B]$: $\mathcal{E}$ applied to half $|\Phi\rangle$ then measure weight 2 stab op. $\rightarrow$ Outside Edge

- Corollary: Any noise model that enters Clifford Polytope via "B-type" facet has tight threshold.

Step 3: Tight Threshold for Depolarizing

- The corollary applies to depolarizing noise.
- We can prove that whenever depolarized R is outside the Clifford polytope, it is outside a "B-type" facet.
- We know that "B-type" facets lead to tight thresholds.
- Since $M = (1 - p)R$, suffices to prove

  $\forall A \in \mathcal{F}_A \cup \mathcal{F}_{A^T}, \quad \exists B \in \mathcal{F}_B$ such that $R \cdot (B - A) \geq 0.$

- Using Clifford symmetry, we just consider a subset of $R \in SO(3)$ without loss of generality.

Step 3: Tight Threshold for Depolarizing

- Using Clifford symmetry, only need to consider R s. t.

  $R \cdot A$ is maximized by $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

  and $\quad -R_{1,2} \geq |R_{i,j}| \qquad (i \in \{1,2,3\}, j \in \{2,3\})$

$$\Rightarrow R \in \left\{ \begin{pmatrix} + & - & + \\ + & + & - \\ + & + & + \end{pmatrix}, \begin{pmatrix} + & - & - \\ + & + & - \\ + & + & + \end{pmatrix}, \begin{pmatrix} + & - & - \\ + & + & - \\ + & - & + \end{pmatrix}, \begin{pmatrix} + & - & + \\ + & - & - \\ + & + & + \end{pmatrix} \right\}.$$

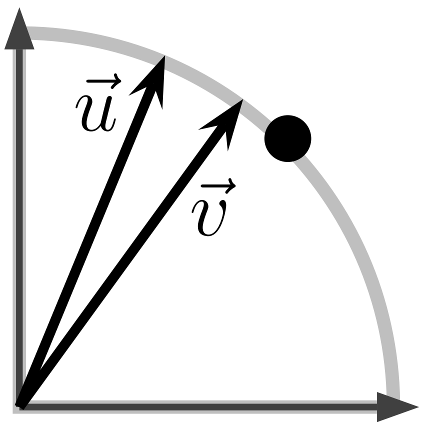
- For such R, we know which $B \in \mathcal{F}_B$ maximizes $R \cdot B$.

$$R \cdot (B - A) = \begin{pmatrix} + & - & \cdot \\ + & \cdot & - \\ + & \cdot & + \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right]$$

Step 3: Tight Threshold for Depolarizing

$$R \cdot (B - A) \geq 0 \quad \Leftrightarrow \begin{pmatrix} + & - & \cdot \\ + & \cdot & - \\ + & \cdot & + \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \geq 0$$

- Define $\vec{u} = (R_{1,1}, R_{1,2})$ and $\vec{v} = (R_{2,3}, R_{3,3})$.
- Note that $\vec{u}$ and $\vec{v}$ have the same Euclidean norm.
- $R \cdot (B - A) \geq 0 \quad \Leftrightarrow \quad ||\vec{v}||_1 - ||\vec{u}||_1 \geq 0$



By assumption:
$|R_{1,2}| \geq |R_{2,3}|, |R_{3,3}|$
and hence
$||\vec{v}||_1 \geq ||\vec{u}||_1$

Open Questions

- Close the gap for ancilla distillation?
- Other noise models (e.g. Non-unital Noise)
- Allow noise to affect stabilizers (Virmani, Plenio arXiv:0810.4340)

Acknowledgement:

This material is based upon work supported

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