Eliminating spurious eigenvalues in the analysis of incompressible fluids and other systems of differential-algebraic equations

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Abstract

We describe a general framework for avoiding *spurious eigenvalues* — unphysical unstable eigenvalues that often occur in hydrodynamic stability problems. In two example problems, we show that when system stability is analyzed numerically using *descriptor notation*, spurious eigenvalues are eliminated. Descriptor notation is a generalized eigenvalue formulation for differntial-algebraic equations that explicitly retains algebraic constraints. We propose that spurious eigenvalues are likely to occur in the analysis of any set of differential-algebraic equations when the algebraic constraints are used to analytically reduce the number of independent variables before the system is approximated numerically. In contrast, the simple and easily generalizable descriptor framework simultaneously solves the differential equations and algebraic constraints and is well-suited to stability analysis in these systems.

Key words: spurious eigenvalue, descriptor, differential algebraic, spectral method, incompressible flow, hydrodynamic stability, generalized eigenvalue, collocation
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1 Introduction

Spurious eigenvalues are unphysical, numerically-computed eigenvalues with large positive real parts that often occur in hydrodynamic stability problems. We propose that these unphysical eigenvalues are possible in the numerical analysis of any set of differential-algebraic equations which is *analytically reduced* – i.e., the algebraic constraints are used to reduce the number of independent variables before the system is approximated using finite difference or spectral collocation methods. An alternative approach to analyzing differential-algebraic equations is the *descriptor* framework, posed as a generalized eigenvalue problem, which explicitly retains the algebraic constraints during the numerical computation of eigenvalues. We reformulate two common hydrodynamic stability problems using descriptor notation and show that this method of computation avoids the spurious eigenvalues generated by other methods. The descriptor formulation is a simple, robust framework for elimi-

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nating spurious eigenvalues that occur in hydrodynamic stability analysis and in the study of other dynamical systems with algebraic constraints. Additionally, this formulation reduces the order of the numerically approximated differential operators and accommodates complex boundary conditions, such as a fluid interacting with a flexible wall.

Spurious eigenvalues are generally found in spectral numerical computations describing incompressible fluids. Researchers have developed special methods to avoid or filter these modes and uncover the true spectrum of the model problem. Perhaps the first description of these unphysical values is given by Gottlieb and Orszag [1]. Many other researchers have encountered similar modes [2,3,4] and developed methods for avoiding [1,5,6,7,8] or filtering [9,10] them. These methods for avoiding spurious modes are specific to very special *clamped* boundary conditions where homogeneous Dirichlet and Neumann conditions hold at the boundaries. In general, it is difficult for a non-expert to determine which method is appropriate for a given problem or how to generalize the methods to describe unusual boundary conditions.

This paper is divided as follows: Section 2 provides background material on eigenanalysis of the Orr-Sommerfeld equation and spectral methods. While notation introduced in this section will be used throughout the paper, the content should be familiar to experts. In Subsection 2.1, operator multiplication is used to derive the Orr-Sommerfeld equation from the incompressible Navier-Stokes equations. Subsection 2.2 reviews Chebyshev collocation schemes for differential operators with special emphasis on inclusion of boundary conditions. Subsection 3.1 discusses the general descriptor framework and Subsection 3.2 develops this method for a particular example, the Orr-Sommerfeld operator. Section 4 discusses how infinite eigenvalues arise in all formulations of the incompressible fluid equations, and explains how the descriptor formulation explicitly accounts for these eigenvalues. We also discuss why methods that include analytical transformations might generate spurious eigenvalues, as well as benefits and drawbacks of descriptor formulations. Finally, Appendix A develops and discusses a descriptor formulation for Gottlieb and Orzag's simple one-dimensional potential flow model [1].

2 Background

2.1 Eigenanalysis and the Orr-Sommerfeld Operator

Eigenvalue analysis is a useful tool for understanding the transition from laminar to turbulent flow – hydrodynamic stability is determined by linearizing around laminar flow and investigating how perturbations change system dynamics. **Linear** stability evaluates the effect of infinitesimal perturbations at long times by analyzing the eigenvalues of the linearized operator [11], while **optimal perturbation** [12,13],**pseudo-spectral** [14], **input-output** [15], and similar methods [16,17,18,19,20] investigate transient growth by analyzing the structure of the linearized operator. In all cases, it is critical to properly resolve the spectrum of the operator.

We propose that spurious eigenvalues are possible in the analysis of any dynamical system where a set of differential-algebraic equations is reduced to a set of differential equations only. Incompressible fluid flow is one such example. The linearized Navier-Stokes equations for a compressible, viscous, isothermal fluid can be written as four partial differential equations that express conservation of mass (one equation involving a time derivative of the pressure) and conservation of momentum (three equations, each involving a time derivative of a component of the velocity) [21]:

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial_t} = -\nabla p + \left(\frac{4\mu}{3} + \eta_B\right) \nabla \left(\nabla \cdot \boldsymbol{v}\right); \tag{1}$$

$$0 = \frac{\partial p}{\partial t} + \rho_0 c^2 \left(\nabla \cdot \boldsymbol{v}\right), \tag{2}$$

where $p = c^2 (\rho - \rho_0)$. Here $\boldsymbol{v}$ is a three-component vector representing the components of the fluid velocity and p is the fluid pressure. As the system approaches the incompressible limit, the partial differential equation expressing conservation of mass density or pressure (Eq. 2) becomes increasingly stiff, and in the limit a differential equation in time is replaced by an algebraic constraint, $\nabla \cdot \boldsymbol{v} = 0$. In this case the pressure can be thought of as a Lagrange multiplier that instantaneously satisfies the divergence constraint. As the equations for pressure become more stiff the corresponding eigenvalues will have larger real parts, and in the limit the eigenvalues will be infinite.

Numerical solutions to the incompressible equations of motion can be determined by rewriting the pressure in terms of the fluid velocities, thereby reducing the set of four differential algebraic equations to two differential equations. In planar channel flow, this results in the Orr-Sommerfeld equation for the wall-normal velocity and the Squire equation for the wall-normal vorticity. There are many derivations of this result [22,23]. To clarify notation, a derivation of the Orr-Sommerfeld operator using operator multiplication is included here.

For a channel flow between rigid walls, the nondimensionalized, linearized Navier-Stokes equations can be written schematically as:

$$\dot{\boldsymbol{v}} = A\boldsymbol{v} + Qp; \tag{3}$$
$$\mathcal{D}\boldsymbol{v} = 0, \tag{4}$$

where Q is the column operator $-\{\partial_x, \partial_y, \partial_z\}'$, and $\mathcal{D}$ is the row operator $\{\partial_x, \partial_y, \partial_z\}$. A no-slip condition at the boundaries requires $\boldsymbol{v} = \boldsymbol{0}$, while there are no explicit boundary conditions on p. The operator A can be written as follows:

$$A = \begin{bmatrix} \frac{\Delta}{R} + U\partial_x & U' & 0 \\ 0 & \frac{\Delta}{R} + U\partial_x & 0 \\ 0 & 0 & \frac{\Delta}{R} + U\partial_x \end{bmatrix}, \tag{5}$$

where R is the Reynolds number and U is the mean flow. Figure 1 shows

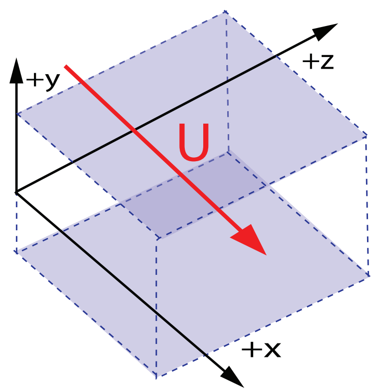

Fig. 1. Schematic diagram of channel flow geometry. The mean flow is in the x-direction,and the system is infinite in the x- and z-directions. Rigid walls bound the flow in the y-direction.

the flow geometry and axis labels. The mean flow is in the x-direction, while the channel walls ensure the flow is non-periodic in the y-direction. To derive the Orr-Sommerfeld equation, we first to rewrite the pressure in terms of the velocities. Because the operator $\mathcal{D}$ commutes with time derivatives, left multiplication by $\mathcal{D}$ on Eq. 3 results in a left hand side which is identically zero. Therefore p and $\boldsymbol{v}$ have the following relationship:

$$\mathcal{D}\left(A\boldsymbol{v}\right) = -\mathcal{D}Qp. \tag{6}$$

A three-by-three matrix is formed from the product of the scalar $\mathcal{D}Q \equiv -\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) \equiv -\Delta$ and the identity matrix (I). Left multiplication by $\mathcal{D}QI$ of Eq. 3 results in the following equation:

$$\mathcal{D}QI\dot{\boldsymbol{v}} = \mathcal{D}QI\left(A\boldsymbol{v}\right) + \mathcal{D}QIQp. \tag{7}$$

Because $\mathcal{D}QI$ commutes with Q the two may be interchanged in the last term of Eq. 7. The pressure p can then be removed from the equation using Eq. 6, resulting in:

$$\Delta I\dot{\boldsymbol{v}} = \Delta I\left(A\boldsymbol{v}\right) + \left(Q\mathcal{D}\right)\left(A\boldsymbol{v}\right), \tag{8}$$

where $Q\mathcal{D}$ is a three-by-three matrix. Rewriting the velocity fields in terms of the wall-normal velocity v_y and vorticity w_y and operating on each equation by Δ^{-1}, results in the following equations [24]:

$$\begin{bmatrix} \dot{v_y} \\ \dot{w_y} \end{bmatrix} = \mathcal{A} \begin{bmatrix} v_y \\ w_y \end{bmatrix}, \tag{9}$$

where

$$\mathcal{A} \equiv \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} = \begin{bmatrix} -\Delta^{-1}U\partial_x\Delta + \Delta^{-1}U''\partial_x + \Delta^{-1}\Delta^2/R & 0 \\ -U'\partial_z & -U\partial_x + \Delta/R \end{bmatrix}. \tag{10}$$

The term $\mathcal{A}_{11}$ is the famous *Orr-Sommerfeld* operator acting on the wall-normal velocity, while $\mathcal{A}_{12}$ and $\mathcal{A}_{22}$ are referred to as the *coupling* and *Squire* operators, respectively. This operator method (Eqs. 6–8) for deriving equations for the wall-normal velocity and vorticity is closely related to the traditional derivation given by Gustavsson and Hultgren [23]. In most two-dimensional flow models the system is assumed to be translation invariant in the z-direction, so the coupling operator is zero and eigenvalues of the Orr-Sommerfeld operator determine the system stability. In order to compare our results with previous studies, we focus here on the stability of the Orr-Sommerfeld operator alone.

The entry $\mathcal{A}_{11}$ contains the operator Δ^2, which includes a fourth-order spatial derivative in the non-periodic y-direction and requires four boundary conditions. Two boundary conditions are simply the no-slip conditions from the original equations, $v_y(\pm 1) = 0$. The remaining two boundary conditions *arise from the divergence constraint*:

$$\left.\frac{\partial v_y}{\partial y}\right|_{y=\pm 1} = -\left(\left.\frac{\partial v_x}{\partial x}\right|_{y=\pm 1} + \left.\frac{\partial v_z}{\partial z}\right|_{y=\pm 1} \right) = 0 + 0, \tag{11}$$

where the last equality holds because v_x and v_z are constant (zero) in the x- and z- directions at the boundary. Therefore, homogeneous Neumann boundary condition on the wall-normal velocity is a direct consequence of the incompressible limit. This has important implications for numerical approximations, as well will discuss in Section 4.

2.2 Spectral methods

Many flows are difficult to treat analytically due to non-periodic boundary conditions. Therefore numerical methods for solving these partial differential equations and eigenvalue problems have been developed and refined. Differentiation matrices, which approximate differential operators acting on functions as matrices acting on vectors, are a particularly useful tool. Finite difference methods track function values at points in physical space, while spectral methods approximate functions as finite series of basis functions, and keep track of the series coefficients. Spectral methods for generating differentiation matrices, such as *Galkerin* or *Tau* methods, implemented in either *collocation* or *basis function* schemes, have several advantages over finite differences and finite elements, including their simplicity and exponential error bounds [25]. One drawback to these methods is that they often generate spurious eigenvalues, which might prevent them from being utilized by researchers who are not experts in numerical methods.

To provide a concrete example in this paper we use spectral collocation with Chebyshev polynomials to discretize infinite-dimensional fields (such as the fluid velocity and pressure in the wall-normal direction). In general, spectral methods approximate functions by a truncated series of N basis polynomials, $f_N(y) = \sum_{j=0}^{N} a_j \phi_j$. We will refer to formulations with operators that act directly on these polynomials as *basis function* schemes. Alternatively, spectral *collocation* utilizes the fact that there is a one-to-one correspondence between the coefficients of that series, a_j, and the values of the function at specially chosen, non-uniform grid points, $f_N(y_j)$. Each function $f(y)$ can be approximated by its values at these special set of points, $f_N(y_j)$, and operators are approximated as matrices that act on these points. The grid points are chosen so that the error associated with the approximation is *evanescent*, or $\mathcal{O}\left((1/N)^N\right)$, which is superior to finite difference methods. A particularly clear introduction to spectral collocation is given by Boyd [26]. For non-periodic, bounded problems, Chebyshev polynomials form a good basis set, and the ij entry of the first derivative matrix, $\mathrm{Cheb}^{(1)}_{ij}$, is given by [26]:

$$\mathrm{Cheb}^{(1)}_{ij} \equiv \begin{cases} (1+2N^2)/6, & i=j=0; \\ -(1+2N^2)/6, & i=j=N; \\ -x_j/\left[2(1-x_j^2)\right], & i=j;\ 0<j<N; \\ (-1)^{i+j} p_i/\left[p_j(x_i-x_j)\right], & i\neq j, \end{cases} \tag{12}$$

where

$$p_0 = p_N = 2, \qquad p_j = 1, \quad j \in (1 \ldots N-1). \tag{13}$$

The second-derivative spectral collocation approximation is given by the square of the first-derivative matrix: $\mathrm{Cheb}^{(2)} = \left(\mathrm{Cheb}^{(1)}\right)^2$.

There are two main methods for applying boundary conditions in spectral methods. The first is basis recombination, where fields are expanded in a series of basis functions that independently satisfy the appropriate boundary conditions [26,25], which are called *Galerkin* schemes. Boundary conditions can also be enforced using boundary bordering, which has two variations. The first variation enforces homogeneous Dirichlet boundary conditions in spectral collocation schemes. In this case, the first and last entries in the vector correspond to the boundary points in physical space, and one simply removes the first and last rows and columns of the differentiation matrix. Note that this reduces the size of the square differentiation matrix by two. A second variation involves using m rows of the matrix to enforce m boundary conditions explicitly, which is often referred to as the *Tau* method. Note that both Tau and the Galkerin methods can be implemented with collocation *and* basis function schemes.

3 The descriptor framework

Descriptor notation is a method which is common in literature on control of dynamical systems that solves generalized eigenvalue problems for sets of differential-algebraic equations [27,28,29]. While we are interested in eliminating spurious eigenvalues that arise in hydrodynamic stability analysis, these unphysical eigenvalues seem likely to appear in other systems of differential equations. Therefore, in Section 3.1 we show how descriptor notation can be applied to a general differential-algebraic system. In Section 3.2 we show that this method eliminates spurious eigenvalues for the incompressible linearized Navier Stokes equations.

3.1 General Descriptor formulation

Descriptor notation was developed in the control theory community to describe and analyze systems of differential-algebraic equations. In descriptor form, the differential time operator is preceded by a square, possibly singular matrix:

$$E\frac{\partial}{\partial t}\phi = A\phi. \tag{14}$$

In descriptor systems, stability is determined by the generalized eigenvalues of the (A, E) system, which is the ratio of the pair (α, β) where $\beta A\boldsymbol{u} = \alpha E\boldsymbol{u}$ for some non-zero vector $\boldsymbol{u}$. While traditional eigenvalues are never infinite, descriptor system can have (many) infinite eigenvalues. If E contains a zero row corresponding to an algebraic constraint, there will be an infinite eigenvalue corresponding to the infinitely fast dynamics of that constraint.

Let us assume that we have a system of differential-algebraic equations for n fields. Let there be m algebraic constraints, and $k = n - m$ equations that contain a differential time operator. Physical systems are often modeled by differential-algebraic equations of this form because algebraic constraints often arise as approximations to differential equations for quickly equilibrating variables.

Let the vector $\boldsymbol{v}$ contain the k fields that are acted upon by the differential time operator, and p contain the remaining m fields. Each of the n fields can be discretized using N points for each field, which results in a system of differential algebraic equations that can be written as follows:

$$\begin{bmatrix} I & 0 \\ & \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{v}} \\ \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ & \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \\ p \end{bmatrix}; \tag{15}$$

$$E\dot{\phi} \equiv A\phi. \tag{16}$$

The discretized operators A and E are square matrices with $(n \times N)^2$ entries. If the original equations are partial differential equations, the operator A contains spatial derivatives that require boundary conditions. Any boundary condition can be incorporated into the numerical solution using boundary bordering in a straightforward manner [26]. For example, if A contains spatial derivatives up to order G for each of the k fields in $\boldsymbol{v}$, then G boundary conditions are required for each field and are enforced by using $(G \times k)$ rows of A as algebraic constraints. We then solve for the generalized eigenvalues of the (A, E) pair. We assume the pair is *regular*, which means that $\det(sE - A)$ is not identically zero for all s. There are many numerical routines that solve for the generalized eigenvalues of regular matrix pairs when one of the matrices is singular (see references in [30]and [31]). A widely available routine is the MATLAB (LAPACK) '**QZ**' algorithm [32]. Note that E contains (mN) zero rows corresponding to the original algebraic constraints and (kG) zero rows corresponding to the boundary bordering constraints, resulting in an (A, E) pair with $(mN + kG)$ infinite generalized eigenvalues and $(kN - kG)$ finite eigenvalues. With descriptor notation, we include the algebraic constraints numerically and solve the system of equations simultaneously.

This is in contrast to other methods where the algebraic constraints are removed analytically. In these methods, the system is analytically converted to a system of equations that contains only fields in v. This process may generate an over-specification of the boundary conditions, as we illustrate with an example. Assume that A_{12} contains a first order spatial derivative and A_{22} is zero. We eliminate p using the method described for the Orr-Sommerfeld operator (Eqs. 3-9), and the resulting system of equations contains a spatial derivative of order $2+G$ acting on the fields in $\boldsymbol{v}$. Because the derivative operator is two orders higher than before, the system requires two new boundary conditions. The only way to determine these extra boundary conditions is to numerically approximate the *algebraic constraints* at the boundary. However, the algebraic constraints were used to eliminate p and were already evaluated at the boundary in the analytical computation. It is therefore not surprising that this method for analyzing differential-algebraic equations generates spurious eigenvalues, as the algebraic constraint at the boundary is approximated in two different ways.

3.2 Example: Two-dimensional incompressible linearized Navier Stokes equations

Spurious eigenvalues arise frequently in the analysis of this Orr-Sommerfeld operator [6,33,4]. While there are several methods for avoiding or filtering these eigenvalues, they are tailored to a very particular problem and not easily generalizable. Our approach to solving the original system of Eqs. 3, 4 is more general. We avoid combining the two equations into a single system and instead write the system using descriptor notation. A similar analysis using descriptor notation was applied previously to incompressible Stokes flow in the context of systems control [34,35]. Here we generalize that framework to eliminate spurious eigenvalues in the model system (Eqs. 3,4), which can be written:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & Q \\ D & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ p \end{bmatrix}; \tag{17}$$

$$E\dot{\phi} \equiv \overline{A}\phi; \tag{18}$$

$$v_y(\pm 1) = v_x(\pm 1) = v_z(\pm 1) = 0. \tag{19}$$

The next step is to discretize the system and impose boundary conditions. In the hydrodynamic channel flow problem, the system is translation invariant in the streamwise (x) and spanwise (z) directions, so we can take a Fourier transform of those variables. In the wall-normal direction (y), the velocities have Dirichlet boundary conditions at the boundaries, $y = \pm 1$, and we must discretize the system of equations in the y-direction. As discussed in Section 2, we discretize the system using Chebyshev collocation, and we must apply boundary conditions to the system of equations. No-slip boundary conditions are enforced for the velocities, but care must be taken to ensure that no boundary conditions are applied to the pressure.

For simplicity, we use boundary bordering to enforce Dirichlet boundary conditions on each component of the velocity. We approximate each component of the velocity and the pressure by a vector of N points. Dirichlet boundary conditions requires that the first and last entry of each velocity vector is zero:

$$\begin{aligned} v_i(y_0) &= v_i(+1) = 0; \quad (20) \\ v_i(y_{(N-1)}) &= v_i(-1) = 0; \quad (21) \\ i &= x, y, z. \end{aligned}$$

This is equivalent to deleting the first and last columns of $\text{Cheb}^{(1)}$ and $\text{Cheb}^{(2)}$ that occur in the operators D and A, respectively, in Eq. 17. Additionally, time derivatives of the velocity evaluated at the boundary are zero $\dot{v}_i(y_0) = \dot{v}_i(y_{N-1}) = 0$, which is equivalent to deleting the corresponding rows of A and Q in Eq. 17. As a result, each component of the velocity is represented by an $N-2$ column vector. Let $M = 3 \times (N-2)$. Then the operator A is approximated by an M-by-M matrix, Q is a matrix that is M rows by N columns, D is N rows by M columns. Note that we have not imposed any boundary conditions on p, which is still a vector of size N. We then solve for the generalized eigenvalues of the $(\overline{A}, E)$ pair. In the parameter range studied, the pair is *regular* ($\det(sE - \overline{A})$ is not identically zero for all s). The numerical results in this paper are generated using the MATLAB (LAPACK) '**QZ**' algorithm [32]. Because E contains N rows that are singular, there are N infinite generalized eigenvalues and M finite eigenvalues. The first three non-infinite eigenvalues are shown in Table 3.2. These three eigenvalues match those calculated using other methods which have been identified as real, physical eigenmodes – there are no spurious eigenvalues.

Collocation Method	$\boldsymbol{\lambda_1}$	$\boldsymbol{\lambda_2}$	$\boldsymbol{\lambda_3}$
Descriptor Chebyshev Tau	+.0037	-0.0348	-0.0350
Traditional Chebyshev Galerkin [36]	+0.0037	-0.0348	-0.0350
Traditional Chebyshev Tau	97.557	85.735	0.037

Table 1

This table compares the first three non-infinite eigenvalues for the Orr-Sommerfeld equation for $\alpha = 1$, $R = 10\,000$ for several collocation schemes with $N = 34$. Weideman and Reddy's algorithm [36] is a Chebyshev Galerkin scheme with clamped boundary conditions, and does not generate spurious eigenvalues. However, the boundary conditions in this scheme can not be generalized to inhomogeneous conditions.

A descriptor formulation can be used for any system of differential algebraic equations. As another example, in Appendix A we formulate Gottlieb and Orzag's one-dimensional potential flow model using descriptor notation and show that this method again avoids spurious eigenvalues.

4 Discussion

How does the descriptor formulation avoid spurious eigenvalues? Descriptor notation ensures that infinitely fast modes will retain formally infinite eigenvalues, even when those eigenvalues are computed numerically. The pressure *should* have infinite eigenvalues because the pressure responds instantaneously to changes in velocity. In the discretized system there are N discretized pressure variables which correspond to N eigenvalues which are *computed* to be formally infinite. Additionally, this method does not explicitly enforce Neumann boundary conditions on the second order differential operator acting on the velocities – instead, the divergence constraint naturally provides this behavior at the boundaries. The descriptor formulation uses only the no-slip boundary conditions and applies them in an intuitive and unambiguous way. Therefore this method has not introduced any unphysical, fast/spurious modes to the M independent discretized velocity variables. Instead, the divergence constraint on the velocities constrains the pressure at the boundaries and implicitly provides the appropriate boundary conditions. This avoids spurious eigenvalues, as evidenced by the values in Table 3.2.

To understand why spurious eigenvalues occur in the traditional Orr-Sommerfeld formulation, recall that in order to rewrite the pressure in terms of the velocities, we must enforce the divergence constraint everywhere, including at the boundaries. In the process we derive an equation with a fourth-order derivative operator acting on v_y, which requires two additional boundary conditions — the Neumann conditions on v_y. As shown by Eq. 11 and the following discussion, the Neumann conditions on the wall-normal velocity at the boundaries are a consequence of evaluating the divergence constraint at boundaries. Therefore this method has applied the same algebraic constraint (which has infinite eigenvalues) two times, although the numerical approximations to the constraints are possibly different each time. Dawkins, Dunbar and Douglass [37] have shown that a particular discretization of a model for a 1-D fluid (see Appendix A) generates matrix pairs with two formally infinite eigenvalues. They also show that other discretization schemes generate spurious eigenvalues which are approximations to the infinite ones. These considerations suggest that spurious eigenvalues in hydrodynamic stability problems are approximations to two infinite eigenvalues that arise when we approximate the divergence constraint two different ways. As discussed in Section 3.1, this phenomena is not specific to homogeneous Neumann conditions or incompressible fluid flow. Spurious eigenvalues are likely to be a general feature of **reduced** differential-algebraic equations, because the algebraic constraint is enforced once to reduce the system of equations and then again to find boundary conditions for the higher order operator.

Utilizing descriptor notation is similar in spirit to several other methods for avoiding spurious eigenvalues which require the algebraic constraints to be discretized separately from the differential equations [1,6,26]. Gottlieb's method utilizes a *shooting* algorithm for determining eigenvalues – his algorithm was developed before efficient matrix methods were available for solving generalized eigenvalue problems with singular matrices. Gardner and Boyd also describe methods where the algebraic constraint is discretized. The descriptor framework generalizes these ideas and presents a simple, systematic method for avoiding unphysical spurious modes by using new and efficient '**QZ**' algorithms.

One area of research where descriptor notation is likely to be extremely promising is the study of a fluid interacting with a compliant boundary. In this system, no-slip conditions at the wall require that the fluid velocity match the wall velocity there. These complicated Dirichlet boundary conditions can be applied directly to the second order differential operators in A. Furthermore, the fluid pressure at the boundary remains as an independent variable in the eigenvalue computation, which is advantageous because the pressure at the boundary influences wall motion. This topic is currently under investigation.

Descriptor notation might be advantageous to many researchers who study stability of differential-algebraic equations. Although using descriptor notation increases the size of the state space and requires computation of generalized eigenvalues for singular matrices, newer versions of the **QZ** algorithm are quite efficient [30,31]. A descriptor formulation does not require inversion of a differential operator, can be adapted to different discretization schemes, and reduces the order of numerically approximated differential operators — which increases resolution for a fixed number of grid points or basis functions. Additionally, boundary conditions can be applied in a straightforward, intuitive way and descriptor notation can be simply extended to problems where boundary conditions are non-trivial. The simplicity and generalizability of the descriptor framework suggest that it is well-suited to stability analysis in many differential-algebraic systems, including but not limited to incompressible fluids.

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A The incompressible limit of a 1D fluid model

Spurious eigenvalues have been studied in greatest depth using the model problem of Gottlieb and Orszag [1]. This is a model for a two-component, one-dimensional fluid flow at low Reynolds number, and is described by the following equations:

$$\frac{\partial \zeta}{\partial t} = \nu \frac{\partial^2 \zeta}{\partial x^2}; \tag{A.1}$$

$$\zeta = \frac{\partial^2 \psi}{\partial x^2}. \tag{A.2}$$

Here ζ is the vorticity and ψ is the stream function defined by $(v_x, v_y) = (-\partial\psi/\partial y, \partial\psi/\partial x)$. Note that the divergence constraint is automatically satisfied because $\nabla \cdot (v_x, v_y) \equiv 0$ equates the mixed partial derivatives of ψ, which is always true for analytic ψ. For a fluid between stationary rigid walls, no-slip conditions on the velocity at the boundary correspond to the following constraints on the stream function:

$$\psi(x = \pm 1, t) = \psi_x(x = \pm 1, t) = 0. \tag{A.3}$$

The usual method used to find the eigenvalues of Eqs. A.1 combines the two equations into a single partial differential equation,

$$\psi_{xxt} = \nu \psi_{xxxx}, \tag{A.4}$$

then inverts the second order differential operator and represents the operators in terms of spectral differentiation matrices,

$$\psi_t = \nu (\partial_{xx})^{-1} \partial_{xxxx} \psi. \tag{A.5}$$

The second order operator is rendered invertible through the application of two boundary conditions. However, there are four boundary conditions on ψ, so there is ambiguity regarding the choice of applied boundary conditions. One choice uses basis recombination so that each of the basis functions individually satisfies all four boundary conditions. Dawkins, Dunbar and Douglass [37] have shown that using this method with Chebyshev basis functions generates spurious eigenvalues. In the same paper they also show that Legendre basis functions generate formally infinite eigenvalues, and the spurious eigenvalues are approximations to these infinite eigenvalues. When the Neumann boundary conditions are incorporated into a polynomial solution, they match the form of the Legendre polynomials. It appears likely that the Legendre polynomials are the 'correct' basis for approximating the algebraic constraints, and therefore exactly recover the infinite eigenvalues that correspond to these constraints. Under this interpretation, the spurious eigenvalues are infinite modes that are poorly approximated by the Chebyshev method.

An alternate method for incorporating boundary conditions is the traditional Chebyshev-Tau method, where boundary bordering is used to replace four terms in the Chebyshev expansion with four algebraic constraints. These algebraic constraints are then used to reduce the total number of equations by four, and the eigenvalues of the resulting system of equations is computed [6]. This reduced set of differential equation has been shown to be equivalent to the basis recombination method described above [37], and generates spurious eigenvalues. Various other approaches to solving the spurious eigenvalue problem involve imposing only the Dirichlet boundary conditions on the second order differential operator [7,5].

Again, we can rewrite the system in Eqs. A.1,A.2 in descriptor form:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\zeta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \nu\partial_{xx} & 0 \\ -I & \partial_{xx} \end{bmatrix} \begin{bmatrix} \zeta \\ \psi \end{bmatrix}; \tag{A.6}$$

$$E \begin{bmatrix} \dot{\zeta} \\ \dot{\psi} \end{bmatrix} \equiv A \begin{bmatrix} \zeta \\ \psi \end{bmatrix}. \tag{A.7}$$

There is a second order operator acting on ψ and four boundary conditions for ψ, while a second order operator acts on ζ and there are no boundary conditions on ζ. This is a similar situation to the Orr-Sommerfeld problem, because all four conditions on ψ come from both the no-slip condition and the divergence constraint (the two are intertwined because we use the stream function ψ). Therefore we expect these four modes to have infinite eigenvalues and impose all the boundary conditions on ψ as algebraic constraints. We replace the Chebyshev series for the derivative at the endpoints by equations that enforce the boundary conditions. The resulting approximation to the bottom row of the matrix Eq. A.6 ($0 = -\zeta + \partial_{xx}\psi$) is:

$$0 = \psi_0; \tag{A.8}$$
$$\mathbf{0} = -\delta_{ij}\zeta_j + \text{Cheb}^{(2)}_{ij}\psi_j, j, i \in (1, N-2); \tag{A.9}$$
$$0 = \psi_{(N-1)}, \tag{A.10}$$

and the approximation to the top row of Eq. A.6, $\partial_t\zeta = \partial_{xx}\zeta$, is:

$$0 = \text{Cheb}^{(1)}_{1j}\psi_j, \quad j \in (1, N-2); \tag{A.11}$$
$$\partial_t\zeta_i = \text{Cheb}^{(2)}_{ij}\zeta_j, \quad i \in (1, N-2), j \in (0, N-1); \tag{A.12}$$
$$0 = \text{Cheb}^{(1)}_{(N-2)j}\psi_j \quad j \in (1, N-2). \tag{A.13}$$

With descriptor notation there are $(N-2)$ infinite eigenvalues corresponding to the $(N-2)$ interior points of the algebraic constraint Eq. A.9. The four

Method	$\boldsymbol{\lambda_1}$	$\boldsymbol{\lambda_2}$	$\boldsymbol{\lambda_3}$
Exact	-9.8696	-20.1907	-39.4784
Descriptor Chebyshev Tau (*collocation*)	-9.8690	-20.1883	-39.4694
Descriptor Chebyshev Tau (*basis fn*)	-9.8696	-20.1907	-39.4784
Traditional Chebyshev Tau (*basis fn*)	56,119	48,515	-9.8696
Traditional Chebyshev Galerkin (*collocation*) [36]	-9.8696	-20.1907	-39.4784

Table A.1

Comparison of the first three non-infinite eigenvalues for the 1D viscous fluid model with several discretization (both collocation and basis function) schemes with $N = 20$.

new boundary conditions correspond to four formally infinite eigenvalues in the numerical spectrum. This final descriptor system contains an E matrix with $(N-2)+4$ rows of zeros, and therefore the system has $N+2$ generalized infinite eigenvalues. If a Chebyshev-Tau method is used to discretize the differential operators, descriptor notation is equivalent to the method suggested by Gottlieb and Orszag to avoid spurious eigenvalues when they first posed this simple model [1], except that we use matrix methods for singular matrices instead of a shooting algorithm to determine eigenvalues. Again, the '**QZ** ' routine in MATLAB is used to compute the spectrum for this system of equations. Table A compares the first three non-infinite numerical eigenvalues to the analytically computed eigenvalues for several different discretization schemes, and confirms that this method generates no spurious eigenvalues.

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