Problems for HW 4

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1 HW3 Problem 1

Suppose that charge density depends only on z: $\rho = \rho(z)$. Thus, all charge is in sheets perpendicular to $\hat{z}$. Also, suppose that the boundary conditions depend upon z only.

a) Write down Laplace's Equation, and show that in 1D, its solution is a linear function:

$$V(z) = A_0 + A_1 z, \tag{1}$$

where $\{A_0, A_1\}$ are constants. Note that in these expressions, z is the field point.

b) Consider a thin sheet of surface charge σ_0, at the "source point" z'. • Find the relations among the constants $\{A_0, A_1\}$ for field points above and below the source point: $z > z'$ and for $z < z'$. (You might call these constants $\{A_{0>}, A_{1>}\}$ and $\{A_{0<}, A_{1<}\}$). • With just one sheet, you may assume that the potential is symmetric across the sheet: $V(-(z-z')) = V(+(z-z'))$.

c) In part b, the charge density of a single sheet can be expressed with a delta-function: $\rho_0(z) = \sigma_0 \delta(z - z')$. • Note also that any given charge distribution $\rho(z)$ that varies only with z can be expressed as a superposition of sheets. And, the solution for $V(z)$ in this case is just the superposition of V's. Give an expression for $V(z)$, given as an integral over z'.

d) As a simple example, consider the charge distribution

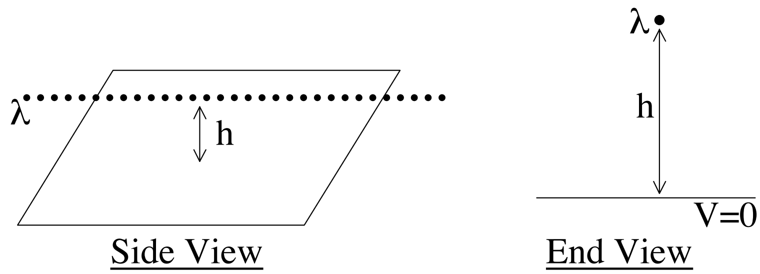
$$\rho(z') = \begin{cases} \rho_0 \sin(\pi z'/a), & -a \le z' \le a \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Find the potential $V(z)$. How could you include the effects of an unknown charge distribution outside $-a \le z \le a$?

In this problem, we found a solution to Laplace's Equation, and then used local boundary conditions to find a solution to Poisson's Equation, for a δ-function charge distribution in a particular geometry. We then used superposition to find a general solution to Poisson's Equation, for an arbitrary charge distribution, in this geometry. This is an extremely common way of finding potential for arbitrary distributions of charge in various geometries. In this simple geometry, it's also possible to solve Poisson's Equation directly by integration of $\rho(z)$. This direct approach isn't always possible in more complicated geometries.

2 HW4 Problem 2

a) An infinite line charge λ resides a distance h from an infinite, grounded plane. Find the potential V_{tot}, due to the line charge and the resulting image charge on the plane.



b) Express your result from part a) in cylindrical coordinates, with the z-axis in the conducting plane parallel to the line charge, and the line charge at $s = h$ and $\phi = 0$. The grounded plane is at $\phi = \pm\pi/2$. Note that this is a solution to Laplace's Equation in a half-space, except at the location of the line charge. (Hint: You may wish to use the law of cosines: $|\vec{a} - \vec{s}| = \sqrt{a^2 + s^2 - 2as\cos(\theta)}$.)

c) Suppose that you know a solution to Laplace's Equation in cylindrical coordinates, $V_1(s, \phi)$. (Assume that the solution does not depend on z.)

• Show that $V_\alpha(s, \phi) = V_1(S, \Phi)$ is also a solution to Laplace's Equation, where α is any constant, and $S(s) = s^\alpha$, $\Phi(\phi) = \alpha\phi$.

For example, if $V_1(s, \phi) = 7s^{1/2}\cos(\phi/2) + 3s^{1/3}\cos(\phi/3)$ is a solution to Laplace's Equation, then so is $V_4(s, \phi) = 7s^2\cos(2\phi) + 3s^{4/3}\cos(4\phi/3)$.

• If the original solution V_1 is valid over a region $s_0 < s < s_1$, $\phi_0 < \phi < \phi_1$, then find the region over which the solution is valid.

d) (*Optional!* Also harder, and somewhat open-ended): Using the technique in part c, you can extend your solutions from part b to a larger range of problems, using values of $\alpha \neq 1$. What do those problems look like? (*Hard!*) Does the charge per unit length on the line charge change, under this transformation? Could you extend other solutions, perhaps involving point charges and their images, using the same technique?

3 Problems from Griffiths

3.1, 3.3, 3.6, 3.10