Solution for HW 7 Problem 2

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1 HW7 Problem 2

This problem asks you to compare the expressions for the multipole expansion in Griffiths and derived in class, given in the lecture notes on "Multipoles" on eres.

a) Consider a very thin shell, with outer radius R, very small thickness $d << R$, and charge density within the shell of $\rho = \rho_0 \cos^2(\theta)$. Here, ρ_0 is a constant. Use the equation on p. 7 of the "Multipole" lecture notes to find b_m, for $m = 0$ and $m = 2$ Argue that the other m's vanish. Use these to find $V(r, \theta)$ outside the shell, at $r > R$.

Evaluate your expression along the x-axis, at $r = x$ and $\theta = \pi/2$. How does the potential vary with x?

1.1 Part (a) Solution

From class, the potential outside an axisymmetric charge distribution $\rho(r, \theta)$ is given by

$$V(r_Q, \theta_Q) = \sum_{\ell=0}^{\infty} b_\ell \, r_Q^{-(\ell+1)} P_\ell(\cos\theta_Q) \tag{1}$$

(Note: "Axisymmetric" means that ρ does not depend on ϕ.) As in class, the subscript "Q" on r_Q and θ_Q indicate that these are the coordinates of the field point. The b_ℓ are constants.

From the "Multipole" notes (available on eres as mentioned above), the b_ℓ are given by:

$$b_\ell = \frac{1}{4\pi\epsilon_0} \int d^3r' (r')^\ell P_\ell(\cos\theta') \rho(r', \theta'). \tag{2}$$

where the integral is over the volume of the charge distribution. Substituting, we find that this is:

$$\begin{aligned} b_\ell &= \frac{1}{4\pi\epsilon_0} \rho_0 \int_0^{2\pi} d\phi \int_{R-d}^{R} dr' (r')^{2+\ell} \int_0^\pi \sin\theta' d\theta' P_\ell(\cos\theta') \cos^2(\theta) && (3) \\ &= \frac{1}{4\pi\epsilon_0} \rho_0 \cdot 2\pi \cdot R^{(\ell+2)} \, d \cdot \int_0^\pi \sin\theta' d\theta' P_\ell(\cos\theta') \cos^2(\theta') && (4) \end{aligned}$$

To do the integral over θ', we can look up the various P_ℓ and do the integrals (or do them using Mathematica, which has a function that gives the P_ℓ; or see the "Note" at the end of this solution for Part a). Here are the results of the integrals: For $\ell = 0$, $P_0 = 1$, and the integral over θ' yields:

$$\int_0^\pi \sin\theta' d\theta' \cdot 1 \cdot \cos^2(\theta') = -\frac{1}{3}\cos^3(\theta')\Big|_0^\pi = \frac{2}{3} \tag{5}$$

For $\ell = 1$, $P_1 = \cos(\theta')$ and the integral over θ' yields:

$$\int_0^\pi \sin\theta' d\theta' \cdot \cos(\theta') \cdot \cos^2(\theta') = -\frac{1}{4}\cos^4(\theta')\Big|_0^\pi = 0 \tag{6}$$

For $\ell = 2$, $P_2 = \frac{1}{2}(3\cos^2(\theta') - 1)$ and the integral over θ' yields:

$$\begin{aligned}\int_0^\pi \sin\theta' d\theta' \cdot \frac{1}{2}\left(3\cos^2(\theta') - 1\right) \cdot \cos^2(\theta') &= \frac{1}{2}\left\{-\frac{3}{5}\cos^5(\theta') + \frac{1}{3}\cos^3(\theta')\right\}\Big|_0^\pi &(7)\\ &= \frac{4}{15} &(8)\end{aligned}$$

Combining the results of integration over θ' in Eqs. 5-8, with the integrations over r' and ϕ' as given by Eq. 4, we obtain b_0 and b_2:

$$\begin{aligned} b_0 &= \frac{1}{4\pi\epsilon_0}\rho_0 2\pi R^2\, d\frac{2}{3} = \frac{1}{4\pi\epsilon_0}2\pi\frac{2}{3}\rho_0 d R^2 &(9)\\ b_2 &= \frac{1}{4\pi\epsilon_0}\rho_0 2\pi R^4\, d\frac{4}{15} = \frac{1}{4\pi\epsilon_0}2\pi\frac{4}{15}\rho_0 d R^4 &(10)\end{aligned}$$

Hmm, now we're asked to show that all the other b_ℓ's are zero. We can't do that by looking up all of the P_ℓ and doing the integrals, in a finite amount of time. But, we know that the P_ℓ are complete, so that $\cos^2(\theta')$ can be expressed as a sum over them, with weights. In fact, if only b_0 and b_2 are nonzero, it must be expressed in terms of P_0 and P_2 only. That's easy enough to show, now that we've looked up P_0 and P_2:

$$\begin{aligned}\cos^2(\theta') &= (2/3)\left\{\frac{1}{2}\left(3\cos^2(\theta') - 1\right)\right\} + (1/3)\cdot 1 &(11)\\ &= (2/3)P_2(\cos\theta') + (1/3)P_0(\cos\theta') &(12)\end{aligned}$$

So, our the angular dependence of our charge distribution can be represented in terms of P_0 and P_2 alone, and none of the other P_ℓ are needed; in fact, they can't contribute.

We can now finish up part a by finding the potential along the x-axis, using Eq. 1. We have:

$$V(r_Q, \theta_Q) = b_0 r_Q^{-1} + b_2 r_Q^{-3}\, P_2(\cos\theta_Q) \tag{13}$$

On the x-axis, $x = r_Q$ and $\theta_Q = \pi/2$. At $\theta_Q = \pi/2$, $P_0 = 1$ and $P_2 = -1/2$. So,

$$V(x) = b_0 \frac{1}{x} \cdot (1) + b_2 \frac{1}{x^3} \cdot \left(-\frac{1}{2}\right) \tag{14}$$

$$= \frac{1}{4\pi\epsilon_0} \left(2\pi \frac{2}{3} \rho_0 d R^2 \ \cdot \frac{1}{x} - 2\pi \frac{2}{15} \rho_0 d R^4 \ \cdot \frac{1}{x^3}\right) \tag{15}$$

Note: If you find it easier to figure out the constants that make P_0 and P_2 add up to $\cos^2\theta'$ in Eqs. 11 and 12, than to do the integrals in Eqs. 5-8, then you can use that fact, instead of doing the integrals in Eqs. 5-8. Just recall the orthogonality fact for Legendre polynomials:

$$\int_{-1}^{1} du\ P_\ell(u)\, P_m(u) = \begin{cases} \frac{2}{2\ell+1}, & \ell = m \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

where for us, $u = \cos\theta'$, $du = +\sin\theta'\ d\theta'$, and the limits of the integral are from 0 to π; this leads to the more useful form that we use more often:

$$\int_0^{\pi} \sin\theta'\, d\theta'\ P_\ell(\cos\theta')\, P_m(\cos\theta') = \begin{cases} \frac{2}{2\ell+1}, & \ell = m \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

These facts are discussed further in the book, and were used in the "Multipole" notes on eres.

So: choose $m = 0$ or 2, multiply both sides by P_m, and integrate over θ'. We find for $m = 0$:

$$\int_0^{\pi} \sin\theta' P_0(\cos\theta') \cos^2\theta' = \int_0^{\pi} \sin\theta' P_0(\cos\theta') \left(\frac{2}{3} P_2(\cos\theta') + \frac{1}{3} P_2(\cos\theta')\right) \tag{18}$$

$$= \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot \frac{2}{2 \cdot 0 + 1} = 2/3 \tag{19}$$

and, for $m = 2$,

$$\int_0^{\pi} \sin\theta' P_2(\cos\theta') \cos^2\theta' = \int_0^{\pi} \sin\theta' P_2(\cos\theta') \left(\frac{2}{3} P_2(\cos\theta') + \frac{1}{3} P_2(\cos\theta')\right) \tag{20}$$

$$= \frac{2}{3} \cdot \frac{2}{2 \cdot 2 + 1} + \frac{1}{3} \cdot 0 = 4/15 \tag{21}$$

These are easier than the integrals in Eqs 5-8!

b) Now consider the same problem from the standpoint of Eq. 3.95. Find the potential for the charge distribution at $\vec{r} = (x, 0, 0)$ using that equation. Note that θ' here measures the angle with respect to the x-axis.To keep things straight, I suggest the notation:

$$\theta_Z \equiv \arccos(z/r) \qquad \phi_Z \equiv \arctan(x/y) \tag{22}$$

$$\theta_X \equiv \arccos(x/r) \qquad \phi_X \equiv \arctan(z/y) \tag{23}$$

In this notation, the charge density is $\rho(r') = \rho_0 \cos^2 \theta_Z$. Show that this approach gives the same result.

1.2 Part (b) Solution

OK, here's Eq 3.95 in Griffiths:

$$V(\vec{r}_Q) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos\theta')\rho(\vec{r}')dV' \tag{24}$$

where I have added the subscript "Q" to denote the field point.

We can just substitute our expression for ρ into this and evaluate. The one problem is that the stated ρ above depends on the angle with respect to the z-axis, θ_Z: $\rho(\vec{r}_Q) = \rho_0 \cos^2 \theta_Z$. To find the potential on the x-axis, as desired, we must integrate over the angle with respect to the x-axis, θ_X. We can use the facts above, and express this in terms of spherical coordinates (θ_X, ϕ_X) which use the x-axis as the reference. Then

$$\begin{aligned} \cos(\theta_z) &= \left(\frac{z}{r}\right) &(25)\\ &= \left(\frac{z}{s}\right) \cdot \left(\frac{s}{r}\right) &(26)\\ &= \cos(\phi_X)\sin(\theta_X) &(27) \end{aligned}$$

Note that $r = \sqrt{x^2 + y^2 + z^2}$. For convenience, I let $s^2 = z^2 + y^2$; this is the distance from the x-axis, analogous to $\sqrt{x^2 + y^2}$ in the more usual spherical coordinates where the z-axis is the reference.

Thus,

$$\rho(\vec{r}_Q) = \rho_0 \sin^2(\theta_X)\cos^2(\phi_X) \tag{28}$$

The integral in Eq. 3.95 then becomes:

$$\begin{aligned} \int dV'(r')^n P_n(\cos\theta')\rho(\vec{r}') &= \int_{R-d}^{R} (r')^2 dr' \int_0^{2\pi} d\phi_X \int_0^{\pi} \sin\theta_X d\theta_X &(29)\\ &\quad \times (r')^n P_n(\cos\theta_X)\rho_0 \sin^2(\theta_X)\cos^2(\phi_X) &(30) \end{aligned}$$

where I've used the fact that for us $\theta' = \theta_X$. The integrals over r' and ϕ_X are easy:

$$\int_{R-d}^{R} (r')^2 dr' (r\prime)^n = d\, R^{(n+2)} \qquad \text{for } d << R \tag{31}$$

$$\int_0^{2\pi} d\phi_X \cos^2(\phi_X) = \frac{1}{2} 2\pi = \pi \tag{32}$$

For the integral over θ_X, we can again do a lot of integrals on Mathematica. However, note that we'd *really* like to express ρ in terms of *cosines* of θ_X. So then,

$$\sin^2(\theta_X) = 1 - \cos^2(\theta_X) \tag{33}$$

so actually, the integrals are for a constant, 1, minus the integrals we did before for $\cos^2\theta$. For $n = 0$ the integral over θ is:

$$\begin{aligned}\int_0^{\pi} \sin\theta_X d\theta_X P_0(\cos\theta_X)\left\{1 - \cos^2(\theta_X)\right\} &= \int_0^{\pi} \sin\theta_X d\theta_X \\ &\quad - \int_0^{\pi} \sin\theta_X d\theta_X\ P_0(\cos\theta_X)\cos^2(\theta_X) \\ &= 2 - 2/3 = 4/3 \end{aligned} \tag{34}$$

where we've used the result of Eq. 5. For $n = 2$, the integral over θ is:

$$\begin{aligned}\int_0^{\pi} \sin\theta_X d\theta_X\ P_2(\cos\theta_X)\left\{1 - \cos^2(\theta_X)\right\} &= -\int_0^{\pi} \sin\theta_X d\theta_X\ P_2(\cos\theta_X)\cos^2(\theta_X) \\ &= -4/15 \end{aligned} \tag{35}$$

where we've used the results of Eq. 8. For all other values of n, we again get zero, by the same argument as in part a.

Putting together the integrals over r', over ϕ_X, and over θ_X, and remembering to include the constant ρ_0, we have for Eq. 3.95:

$$V(\vec{r}_Q) = \frac{1}{4\pi\epsilon_0}\left\{\frac{1}{r_Q} \cdot d\, R^2 \cdot \pi \cdot \rho_0(4/3) + \frac{1}{r_Q^3} \cdot d\, R^4 \cdot \pi \cdot \rho_0(-4/15)\right\} \tag{36}$$

$$= \frac{1}{4\pi\epsilon_0}\left\{\frac{4\pi}{3} d\, R^2 \rho_0 \frac{1}{z} - \frac{4\pi}{15} d\, R^4 \rho_0 \frac{1}{z^3}\right\} \tag{37}$$

where we have used the fact that $r_Q = x$ along the x-axis. This is the same as Eq. 15, as desired.

c) Compare Eq. 3.95 in Griffiths with the corresponding expressions in the lecture notes: the expression for $V(r, \theta)$ outside, at $r > R$, on p. 4 and the expression for b_m at the top of p. 7. The expressions look pretty similar. But, they are not identical.

Note that the expression in the notes includes θ twice: once as a variable of integration in calculating b_m (where it is the coordinate of the source point and probably should be called θ'), and once as an argument in the general expression for $V(r, \theta)$ (where it is the coordinate of the field point and might be called θ_Q). The expression in Griffiths includes θ' only, as a variable of integration: it is the coordinate of the source point.

Also, in class we assumed that the charge distribution $\rho(r', \theta')$ is axisymmetric: it may depend on r' and θ', but is independent of ϕ'. Griffiths claims that Eq. 3.95 holds for an arbitrary localized charge distribution (see the remarks just above Eq. 3.91). And, in Eqs. 3.92 through 3.94 he treats the charge distribution as a superposition of point sources, and uses HW5 Problem 1 to expand this as a series of Legendre polynomials in θ'. What determines the axis for measurement of θ' here? Moreover, the argument of V in Eq. 3.95 is $\vec{r}$: this suggests that Eq. 3.95 holds at any point in space.

How can Griffiths' expression involve fewer variables, yet be more general? Explain in a brief paragraph. (Hint: Try the preceding parts before you commit to paper.)

1.3 Part (c) Solution

The answer is straightforward after working through the preceding parts a and b. To use Griffith's expression, you need to do a different integral for each choice of field point, $\vec{r}_Q$ (unless you can re-use the integrals done previously, as in this example; or if you are comparing field points along one axis leading out from the origin). So, Griffith's expression need not involve θ_Q or ϕ_Q. It also doesn't need for the charge distribution to be axisymmetric; this is less mysterious than it seems, since the expression is good only for one direction of field point.

For the expression derived in the notes, it is good for any field point outside the charge distribution. So, the expression for the potential it must involve θ_Q as well as r_Q. It *does* require that the charge distribution be axisymmetric; that's why the expression need not involve ϕ_Q. Jackson gives the more general expression, which involves ϕ_Q and requires spherical harmonics. Note that this expression requires doing the integrals only once, no matter what the field point is.