A Charge Within a Conductor

Consider the situation discussed in class: a positive ion situated inside a conductor. Griffiths doesn't consider this problem in detail (although HW3 problem 2.46 provides part of the answer). But, it seemed to draw interest in class. Therefore, I discuss it in these notes.

For definiteness, consider a sphere made of silver (an excellent conductor), with radius 1 cm. The sphere initially has zero net electric charge: it has an equal number of electrons, and (in silver nuclei) protons. Suppose that a positive uranium ion ($Q = +92e$) is placed at the center of the sphere. The ion is locked to the crystalline lattice: it is not free to move.

Electrons near the uranium ion will form a neutralizing cloud around it. But, the charge may not be completely neutralized: for example, the thermal motions of electrons might keep them from being captured by the stationary ion. The captured electrons will reduce the charge of the ion, to $Q < 92e$. But also, free electrons will be deflected toward the atom as they pass by, forming a neutralizing cloud of negative charge around it. That cloud will have a radius of only a few nanometers. Outside that cloud, the electric field will be zero: so the total negative charge from captured plus deflected electrons precisely cancels the $+Q$ of the uranium ion.

The electrons cancelling $+Q$ will be removed from the "sea" of electrons that neutralize the silver sphere. The sphere retains net charge $+Q$; but, that charge can be moved anywhere, by shifts of the "sea". Because any excess within the sphere will be cancelled by yet another electron cloud, the net positive charge must reside on the surface of the sphere: the electrons will withdraw a bit from that surface, leaving an (extremely narrow) positive charge density on the surface. Similar cancelling effects will eliminate any buildup of charge in one spot or another, leaving a surface charge $\sigma = 92e/4\pi(0.01 \text{ m})^2$.

The potential of the surrounding electron cloud can't be expressed exactly (and quantum-mechanical effects are important close to the ion). However, the spatial scale of the cloud is given by the Debye length:

$$\lambda = \lambda_D \equiv \sqrt{\frac{k_B T}{Ne^2/\epsilon_0 \epsilon_R}}. \tag{1}$$

Here, k_B is the Boltzmann constant, T is temperature, N is free electrons per volume (1 per atom for silver), e is electron charge, ϵ_0 is a familiar constant, and ϵ_R is the relative dielectric constant. (I can't find a good value for silver but I think it's between 1 and 10). It's fun to put in values and find the size. It's also fun to try to express this as a ratio of thermal energy to the electrostatic energy of the charge distribution: how does λ_D enter?

The Debye length first came up in studies of ions in solution. It's still important for that; DNA, for example, has strong charges that interact with those on other molecules over a Debye length (in water, in this case). It's of basic importance in plasmas (such as in the Sun and interstellar or intergalactic space), and in semiconductor materials (where ions are frequently implanted so as to add or subtract charge carriers).

So where did Problem 2.46 come in? If the motions of the electrons are drawn from a thermal Maxwell distribution (so that we can ignore details like Fermi-Dirac statistics), then Poisson's Equation is hard to solve in the region near the ion. Also quantum-mechanical effects may come in. However, if we aren't too close to the ion, the potential in 2.46 is the approximate form of the potential V.