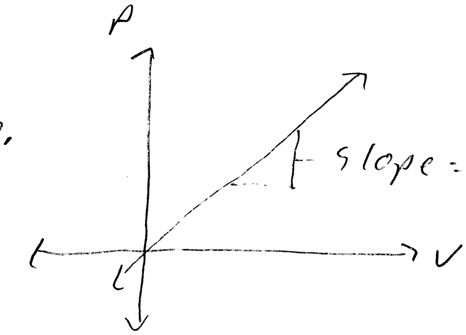
Carter

2.3 a.

$$P = Av$$
$$P_1 = Av_1$$
$$P_1 v_1 = RT_1$$
$$v_1 = \frac{RT_1}{P_1}$$
$$P_1 = \frac{ART_1}{P_1}$$
$$A = \frac{P_1^2}{RT_1}$$

b.



slope $= \frac{P_1^2}{RT_1}$

c.

$$T = \frac{Pv}{R} = \frac{Av^2}{R}$$
$$T_1 = \frac{Av_1^2}{R}$$
$$T \propto v^2; \quad v = 2v_1$$
$$T = \frac{A}{R} \cdot 4v_1^2 = 4T_1 = 800 \text{ K}$$

A-1 a. $dz = 2x \ln y \, dx + (x^2/y) \, dy$

$$A = 2x \ln(y) \qquad B = x^2/y$$
$$\frac{\partial A}{\partial y} = \frac{2x}{y} \qquad \frac{\partial B}{\partial x} = \frac{2x}{y}$$

✓ Exact

$$\frac{\partial f}{\partial x} = A = 2x \ln y \qquad \frac{\partial f}{\partial y} = B = \frac{x^2}{y}$$
$$f = x^2 \ln y + u(x) \qquad f = x^2 \ln y + v(x)$$
$$f = x^2 \ln y + C$$

b. $dz = (y-1)\,dx + (x-3)\,dy$

$A = y-1 \qquad B = x-3$

$\frac{\partial A}{\partial y} = 1 \qquad \frac{\partial B}{\partial x} = 1 \qquad$ Exact

$\frac{\partial f}{\partial x} = A = y-1 \qquad \frac{\partial f}{\partial y} = B = x-3$

$f = xy - x + u(y) \qquad f = xy - 3y + u(x)$

$u(y) = -3y \qquad u(x) = -x$

$f = xy - x - 3y + C$

c. $dz = (2y^3 - 3x)\,dx - 4xy\,dy$

$A = 2y^3 - 3x \qquad B = -4xy$

$\frac{\partial A}{\partial y} = 6y^2 \qquad \frac{\partial B}{\partial x} = -4y \qquad$ Inexact.

A-4 a. $dz = 2y\,dx + 3x\,dy$

$\frac{\partial A}{\partial y} = 2 \qquad \frac{\partial B}{\partial x} = 3 \qquad$ Inexact

$dw = 2yu\,dx + 3xu\,dy$

Assume u is a function of x only (see Carter pg. 397)

$$u = \exp\left[\int \frac{1}{3x}(2-3)\, dx\right] = \exp\left[\int -\frac{1}{3x}\, dx\right]$$

$$= \exp\left[-\frac{1}{3}\ln x + c\right] = \exp\left[\ln x^{-1/3} + c\right]$$ Choose $c=0$.

$$u = x^{-1/3}$$

$$dw = 2yx^{-1/3}\, dx + 3x^{2/3}\, dy$$

$$\frac{\partial A}{\partial y} = 2x^{-1/3} \qquad \frac{\partial B}{\partial x} = 2x^{-1/3} \quad ✓$$

b. $dz = y\, dx + (2x - y^2)\, dy$

$$\frac{\partial A}{\partial y} = 1 \qquad \frac{\partial B}{\partial x} = 2$$ Inexact

$$dw = yu\, dx + u(2x-y^2)\, dy$$

$$\frac{\partial}{\partial y}(yu) = \frac{\partial}{\partial x}(u(2x-y^2))$$

$$u + y\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}(2x-y^2) + u \cdot 2$$

Easier if $u = u(y)$

$$u + y\frac{\partial u}{\partial y} = 2u \qquad y\frac{\partial u}{\partial y} = u \qquad \frac{1}{u}\, du = \frac{1}{y}\, dy$$

$$\Rightarrow u = y.$$

$$dw = y^2\, dx + y(2x-y^2)\, dy$$

$$\frac{\partial A}{\partial y} = 2y \qquad \frac{\partial B}{\partial x} = 2y \quad ✓$$

c. $dz = (x^4+y^4)\,dx - xy^3\,dy$

$$\frac{\partial A}{\partial y} = 4y^3 \qquad \frac{\partial B}{\partial x} = -y^3 \qquad \text{Inexact}$$

$$dw = u(x^4+y^4)\,dx - uxy^3\,dy$$

$$\frac{\partial}{\partial y}\left(u(x^4+y^4)\right) = \frac{\partial}{\partial x}\left(-uxy^3\right)$$

$$\frac{\partial u}{\partial y}(x^4+y^4) + u \cdot 4y^3 = -\frac{\partial u}{\partial x}\cdot xy^3 - uy^3$$

Let $u = u(x)$

$$4\cancel{y^3}u = -x\cancel{y^3}\frac{\partial u}{\partial x} - u\cancel{y^3} \qquad \text{Can cancel } y^3\text{s ...}$$

$$-4u = x\frac{\partial u}{\partial x} + u$$

$$x\frac{\partial u}{\partial x} = -5u \qquad \frac{\partial u}{5u} = \frac{-\partial x}{x}$$

Integrating both sides:

$$\frac{1}{5}\ln u = -\ln x$$

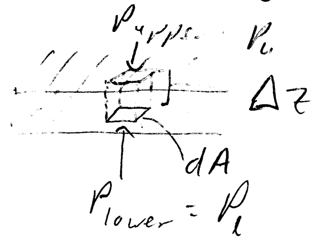
$$u = x^{-5}$$

$$dw = x^{-5}(x^4+y^4)\,dx - x^{-4}y^3\,dy$$

$$\frac{\partial A}{\partial y} = 4x^{-5}y^3 \qquad \frac{\partial B}{\partial x} = 4x^{-5}y^3 \quad \checkmark$$

Schroeder 1.16

a.



$$P_l \cancel{dA} - P_u \cancel{dA} = g \rho \cancel{dA} \Delta z$$

where ρ is the density of air.

$$-(P_u - P_l) = g \rho \Delta z$$

$$-\frac{\Delta P}{\Delta z} = g\rho \qquad \frac{dP}{dz} = -g\rho$$

b, $\rho = \frac{\text{mass}}{\text{Volume}} = \frac{n \cdot m \cdot N_A}{V}$, But $PV = nRT$.

$\left(\text{or } \frac{P}{RT} = \frac{n}{V}\right)$

So: $\rho = \frac{P}{RT} \cdot m \cdot N_A$

using $k = \frac{R}{N_A}$; $\frac{1}{k} = \frac{N_A}{R}$, ρ becomes

$$\rho = \frac{mP}{kT} \qquad \frac{dP}{dz} = -\frac{gmP}{kT}$$

c. $$\frac{1}{P} dP = \frac{-gm}{kT} dz$$

Integrating both sides:

$$\ln P = -\frac{gm}{kT} z + C$$

$$P(z) = e^C \cdot e^{-mgz/kT} \qquad P(z=0) = P_0 \Rightarrow e^C = P_0.$$

$$P(z) = P_0 e^{-mgz/kT} \qquad \rho = \frac{mP}{kT} = \frac{mP_0}{kT} e^{-mgz/kT}$$

d. $m = 0.78 \cdot 14 \cdot 2 + 0.21 \cdot 16 \cdot 2 + 0.01 \cdot 39.95$

$m = 28.96 \text{ amu} = \frac{28.96}{N_A} g = \frac{0.02896}{N_A} kg$

$P(1430) = 1 atm \cdot \exp\left[-mg \cdot 1430m / kT\right]$; $g = 9.8 \, m/s^2$

Remember, we need m in kg.

$= 1atm \cdot \exp\left[\frac{-0.02896 \, g \cdot 1430m}{N_A kT}\right]$

$= 1atm \cdot \exp\left[\frac{-0.02896 \cdot g \cdot 1430m}{RT}\right]$ (using $k = \frac{R}{N_A}$)

Assume a reasonable T (I use T = 280 K)

$\frac{0.02896 \cdot g}{RT} = 1.22 \cdot 10^{-4} \, m^{-1}$

$P(1430) = 1 \cdot e^{-1.22 \cdot 10^{-4} \cdot 1430}$

$P(1430) = 0.84 \text{ atm}$

$P(3090) = 1atm \cdot e^{-1.22 \cdot 10^{-4} \cdot 3090}$

$= 0.686 \text{ atm}$

$P(4420) = 1 atm \cdot e^{-1.22 \cdot 10^{-4} \cdot 4420}$

$= 0.583 \text{ atm}$

$P(8850) = 1atm \cdot e^{-1.22 \cdot 10^{-4} \cdot 8850}$

$= 0.34 \text{ atm}$

Schroeder

1.17

a. $PV = nRT\left(1 + \frac{B(T)}{(V/n)}\right)$

$\frac{V}{n} = \frac{RT}{P}\left(1 + \frac{B(T)}{V/n}\right)$ For the purpose of finding V/n, we may assume this term is small.

$\frac{V}{n} \approx \frac{RT}{P}$

$\frac{B(T)}{V/n} \approx \frac{B(T)}{\frac{RT}{P}} = \frac{P}{RT} \cdot B(T)$; $P = 1\text{ atm} = 1.013 \cdot 10^5 \text{ Pa}$

$\frac{B(T_1)}{V/n} = \frac{P B_1}{RT_1} = \frac{1.013 \cdot 10^5 \text{ Pa} \cdot -160 \cdot 10^{-6} \text{ m}^3/\text{mol}}{8.315 \frac{\text{J}}{\text{mol} \cdot \text{K}} \cdot 100 \text{ K}}$

$= -0.0195$

$\frac{B(T_2)}{V/n} = \frac{P}{RT_2} B_2 = 12{,}183 \cdot \frac{-35 \cdot 10^{-6}}{200} = -0.0021$

$\frac{B(T_3)}{V/n} = \frac{P}{RT_3} B_3 = 12{,}183 \cdot \frac{-4.2 \cdot 10^{-6}}{300} = -0.00017$

$\frac{B(T_4)}{V/n} = \frac{P B_4}{RT_4} = 0.00027$

$\frac{B(T_5)}{V/n} = \frac{P B_5}{RT_5} = 0.00041$

$\frac{B(T_6)}{V/n} = \frac{P B_6}{RT_6} = 0.00043$

b. Forces between molecules can be either attractive or repulsive. Attractive forces (e.g. dipole-dipole interactions, Van der Waals forces) come into play when molecules are close but not touching. However, at very close distances, repulsive forces dominate. At low T, molecules spend more time at intermediate range, so the attractive forces have a chance to work. At high T, interactions tend to happen more quickly. Molecules get close, collide, then get out. The repulsive forces dominate.

c. $$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT$$

$$\left(PV + \frac{an^2}{V}\right)\left(1 - \frac{nb}{V}\right) = nRT$$

$$PV + \frac{an^2}{V} = nRT\left(1 - \frac{nb}{V}\right)^{-1}$$

$$\left(1 - \frac{nb}{V}\right)^{-1} \approx 1 + \frac{nb}{V} + \frac{1}{2} \cdot -1(-2)\left(\frac{nb}{V}\right)^2 + \ldots$$

$$\approx 1 + \frac{nb}{V} + \left(\frac{nb}{V}\right)^2$$

Note that $\frac{an^2}{V} = \frac{an^2}{nRTV} \cdot nRT$,

$$PV = nRT\left[\left(1 - \frac{nb}{V}\right)^{-1} - \frac{an^2}{nRTV}\right]$$

$$\approx nRT\left[1 + \frac{nb}{V} + \left(\frac{nb}{V}\right)^2 + \ldots - \frac{an^{\cancel{2}}}{\cancel{n}RTV}\right]$$

$$\approx nRT\left[1 + \frac{b}{(V/n)} + \frac{b^2}{(V/n)^2} + \ldots - \frac{a}{RT(V/n)}\right]$$

$$\Rightarrow \quad B(T) \approx b - \frac{a}{RT} \qquad C(T) \approx b^2$$

Additional Problems #1

a. $I = \int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy$

i. $I = \int_0^1 2xy\,dx + \int_0^1 x^2\,dy \quad ; \; y = x$

$$I = \int_0^1 2x^2\,dx + \int_0^1 y^2\,dy$$

$$= \left[2\frac{x^3}{3}\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{2}{3} + \frac{1}{3} = 1$$

ii. $I = \int_0^1 2xy\,dx + \int_0^1 x^2\,dy \quad ; \; y = x^2$

$$= \int_0^1 2x^3\,dx + \int_0^1 y\,dy$$

$$= \left[\frac{2x^4}{4}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

iii. $I = \int_0^1 2xy\,dx + \int_0^1 x^2\,dy \quad ; \; y = x^3$

$$I = \int_0^1 2x^4\,dx + \int_0^1 y^{2/3}\,dy$$

$$= \left[2\frac{x^5}{5}\right]_0^1 + \left[\frac{3}{5}y^{5/3}\right]_0^1$$

$$= \frac{2}{5} + \frac{3}{5} = 1$$

iv. $I = \int_0^1 2xy\,dx + \int_0^1 x^2\,dy \;;\; y = \begin{cases} 0; & 0 \le x < 1 \\ x = 1 & @\ x = 1 \end{cases}$

$= \int_0^1 0\,dx + \int_0^1 1\,dy$

$= 0 + 1 = 1$

b. $A = 2xy, \quad B = x^2$

$\frac{\partial A}{\partial y} = 2x \qquad \frac{\partial B}{\partial x} = 2x \quad \checkmark$ Exact.

c. $\frac{\partial f}{\partial x} = A = 2xy \qquad \frac{\partial f}{\partial y} = B = x^2$

$\Longrightarrow$

$f = x^2 y + u(y) \qquad f = x^2 y + v(x)$

Evidently

$f = x^2 y + C$ (C is a constant)

d. $f(1,1) = 1^2 \cdot 1 + C$

$f(0,0) = 0 + C$

$f(1,1) - f(0,0) = 1 \quad \checkmark$

All exact differentials have this property.

Additional Problems # 2

$V = \frac{4}{3}\pi abc$ Dynamic Variables: $a, c, \ldots$

we have b & S constant. Since we don't have a nice expression for b and we're not differentiating with respect to it, forget it. Regard it as a constant and say

$$S = S(a,c) \longleftrightarrow a = a(S,c) \longleftrightarrow c = c(a,S)$$

We want

1. $$\left.\frac{\partial V}{\partial a}\right|_S = \frac{4}{3}\pi b \left.\frac{\partial}{\partial a}(ac)\right|_S = \frac{4}{3}\pi b\left[c + a\left.\frac{\partial c}{\partial a}\right|_S\right]$$

Is a function of a

Using the cyclic relation:

$$\left.\frac{\partial c}{\partial a}\right|_S \cdot \left.\frac{\partial a}{\partial S}\right|_c \cdot \left.\frac{\partial S}{\partial c}\right|_a = -1$$

$$\left.\frac{\partial c}{\partial a}\right|_S = \frac{-1}{\left.\frac{\partial a}{\partial S}\right|_c \cdot \left.\frac{\partial S}{\partial c}\right|_a}$$

Using the inverse relation: $\left.\frac{\partial a}{\partial S}\right|_c = \frac{1}{\left.\frac{\partial S}{\partial a}\right|_c}$:

$$\left.\frac{\partial c}{\partial a}\right|_S = -\frac{\left.\frac{\partial S}{\partial a}\right|_c}{\left.\frac{\partial S}{\partial c}\right|_a}$$

Since we already have a (relatively) simple expression for S in terms of a, b, c, this makes our lives easier

$$\frac{\partial S}{\partial a}\bigg|_c = \frac{\partial}{\partial a}\bigg|_c 4\pi \left(\frac{a^p b^p + a^p c^p + b^p c^p}{3}\right)^{1/p}$$

$$= \frac{4\pi}{p}\left(\frac{a^p b^p + a^p c^p + b^p c^p}{3}\right)^{\frac{1}{p}-1} \cdot \left(p a^{p-1} b^p + p a^{p-1} c^p\right)$$

$$\frac{\partial S}{\partial c}\bigg|_a = \frac{4\pi}{p}\left(\frac{a^p b^p + a^p c^p + b^p c^p}{3}\right)^{1/p-1} \cdot \left(p a^p c^{p-1} + p b^p c^{p-1}\right)$$

$$\frac{\partial c}{\partial a}\bigg|_S = \frac{-\frac{\partial S}{\partial a}\big|_c}{\frac{\partial S}{\partial c}\big|_a} = \frac{-\left(\cancel{p} a^{p-1} b^p + \cancel{p} a^{p-1} c^p\right)}{\left(\cancel{p} a^p c^{p-1} + \cancel{p} b^p c^{p-1}\right)} = \frac{-a^{p-1}\left(b^p + c^p\right)}{c^{p-1}\left(a^p + b^p\right)}$$

Getting c as a function of a is merely algebra:

$$S = 4\pi\left(\frac{a^p b^p + a^p c^p + b^p c^p}{3}\right)^{1/p}$$

$$3\left(\frac{S}{4\pi}\right)^p - a^p b^p = c^p\left(a^p + b^p\right)$$

$$c = \left(\frac{3\left(\frac{S}{4\pi}\right)^p - a^p b^p}{a^p + b^p}\right)^{1/p}$$

At this point, simply substitute c & $\frac{\partial c}{\partial a}\big|_S$ into 1.