Homework #5 Solutions

Schroeder Problems 2.2, 2.3, 2.5, 2.8, 2.16, 2.18, 2.22, 2.24, 2.33, 2.37

X = not graded

✓ = graded

2.2 a. $2^{20} = 1{,}048{,}576$

b. That is simply one possible microstate.

$$P = \frac{1}{2^{20}}$$

c. $$\binom{20}{8} \cdot \frac{1}{2^{20}} = P$$

$$P = 12.0\%$$

2.3 a. $2^{50} = 1.13 \cdot 10^{15}$

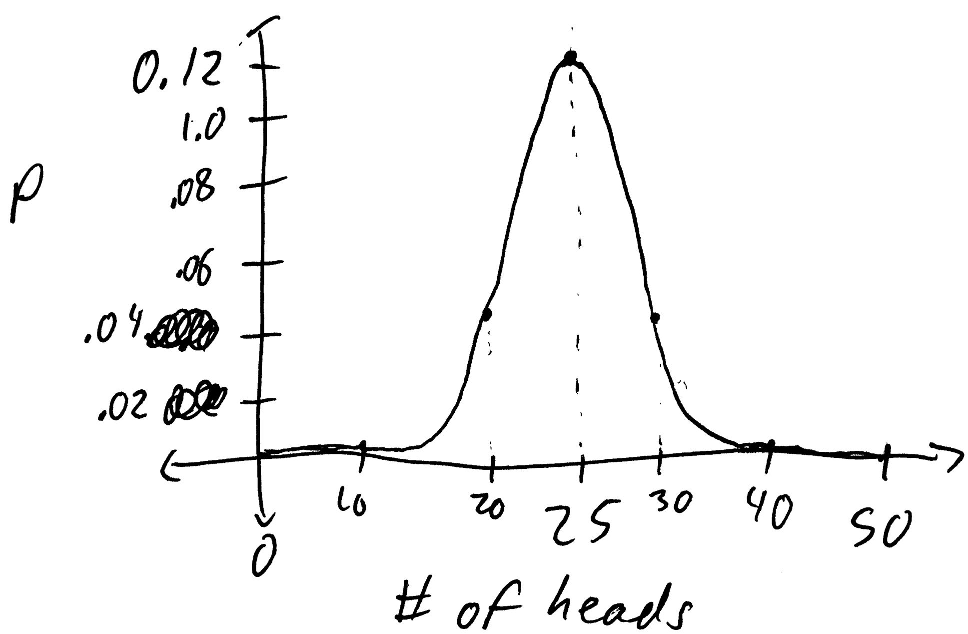
b. $\binom{50}{25} = \frac{50!}{25!\,25!} = 1.26 \cdot 10^{14}$

c. $p = \binom{50}{25} \cdot \frac{1}{2^{50}} = 0.112$

d. $p = \binom{50}{30} \cdot \frac{1}{2^{50}} = 0.042$

e. $p = \binom{50}{40} \cdot \frac{1}{2^{50}} = 9.1 \cdot 10^{-6}$

f. $p = \frac{1}{2^{50}} = 8.88 \cdot 10^{-16}$



2.5 a. $N=3, q=4$

400	310	031	022	211
040	301	103	202	121
004	130	013	220	112

15 total

$$\binom{4+3-1}{4} = \frac{6!}{4!\,2!} = 15 \checkmark$$

c. $N=3, Q=6$ Accidentally did c. first :)

600	501	015	042	141	033	132
060	150	420	204	114	321	213
006	051	402	024	330	312	123
510	105	240	411	303	231	222

28 total

$$\binom{6+3-1}{6} = \frac{8!}{6!\,2!} = 28 \checkmark$$

b. $N=3, Q=5$

500	410	041	320	032	311	221
050	401	104	302	203	131	212
005	140	014	230	023	113	122

21 total

$$\binom{5+3-1}{5} = \frac{7!}{5!\,2!} = 21 \checkmark$$

d. $N=4, q=2$

2000 0020 1100 1001 0101

0200 0002 1010 0110 0011

10 total

$$\binom{2+4-1}{2} = \frac{5!}{2!3!} = 10 \checkmark$$

e. $N=4, q=3$

3000 2100 0210 0021 1110

0300 2010 0201 1002 1101

0030 2001 1020 0102 1011

0003 1200 0120 0012 0111

20 total

$$\binom{3+4-1}{3} = \frac{6!}{3!3!} = 20 \checkmark$$

f. $N=1$: Everything must be in only one oscillator $\Rightarrow$ only 1 microstate

$$\binom{q+1-1}{q} = \frac{q!}{q!} = 1$$

g. Only 1 unit of energy to distribute among N oscillators $\Rightarrow$ N places to put it. N possible microstates

$$\Omega(N,1) = \binom{1+N-1}{1} = \frac{N!}{1!\cdot(N-1)!} = N \checkmark$$

2.8 a. Possible Macrostates:

A	B
0	20 units of energy
1	19
2	18
......	
20	0

21 total

b. Regard the system as a combined system containing 20 oscillators among whom 20 units of energy are divided: $\Omega = \binom{20+20-1}{20} = \boxed{6.89 \cdot 10^{10}}$

c. $\Omega_A = \binom{20+10-1}{20} = 1.00 \cdot 10^{7}$

$\Omega_B = 1$ $\qquad P = \dfrac{\Omega_A \Omega_B}{\Omega_{Total}}$

$= \dfrac{1.00 \times 10^{7}}{6.89 \times 10^{10}} = 1.45 \cdot 10^{-4}$

d. $\Omega_A = \Omega_B = \binom{10+10-1}{10}$

$\Omega_A \Omega_B = \Omega_A^2 = 8.534 \times 10^{9}$

$$P = \frac{\Omega_A \cdot \Omega_B}{\Omega_{total}} = \frac{8.\cancel{8}534 \cdot 10^9}{6.89 \cdot 10^{10}} = 0.124$$

e. Evolution from a less likely state to a more likely state (e.g. starting with all the energy in A and shifting such that half is in A & half is in B) is "irreversible" in the sense that the reverse is extremely unlikely to happen. It is extremely unlikely for the system to start with ~~the~~ evenly distributed energy (half in A, half in B) and spontaneously evolve to a state where all the energy is in A or B.

2.16

a. $$P = \frac{\Omega(500)}{2^{1000}}$$

$$\Omega(500) = \binom{1000}{500} = \frac{1000!}{(500!)^2} \approx \frac{1000^{1000} e^{-1000}\sqrt{2\pi\cdot 1000}}{\left(500^{500} e^{-500}\sqrt{2\pi\cdot 500}\right)^2}$$

$$= \frac{2^{1000}}{\sqrt{500\pi}}$$

$$P = \frac{\Omega}{2^{1000}} = \frac{1}{\sqrt{500\pi}} = 0.025$$

b. $$P = \frac{\Omega(600)}{2^{1000}}$$

$$\Omega(600) = \frac{1000!}{600!\,400!} \approx \frac{1000^{1000} e^{-1000}\sqrt{2\pi\cdot 1000}}{600^{600} e^{-600}\sqrt{2\pi\cdot 600}\cdot 400^{400} e^{-400}\sqrt{2\pi\cdot 400}}$$

$$= \frac{1000^{1000}}{600^{600}\, 400^{400}\sqrt{480\pi}}$$

$$P = \frac{\Omega}{2^{1000}} = \frac{500^{1000}}{600^{600}\, 400^{400}\sqrt{480\pi}} = \left(\frac{5}{6}\right)^{600}\left(\frac{5}{4}\right)^{400}\cdot\frac{1}{\sqrt{480\pi}}$$

$$= 4.6\cdot 10^{-11}$$

2.18 (1.) $(N-1)! = \frac{N!}{N}$, since $N! = N\cdot(N-1)\cdot(N-2)\cdots$

So $(q+N-1)! = \frac{(q+N)!}{q+N}$

$$\Omega(N,q) = \binom{q+N-1}{q} = \frac{(q+N-1)!}{q!\,(N-1)!} = \frac{(q+N)!}{q!\,N!}\cdot\frac{N}{q+N}$$

Stirling's approximation on Ω yields:

$$\Omega \approx \frac{(q+N)^{q+N} e^{-(q+N)}\sqrt{2\pi(q+N)}}{q^q e^{-q}\sqrt{2\pi q}\; N^N e^{-N}\sqrt{2\pi N}}\cdot\frac{N}{q+N}$$

$$= \frac{(q+N)^{q+N}}{q^q N^N}\sqrt{\frac{N}{2\pi q(q+N)}}$$

Note that $(q+N)^{q+N} = (q+N)^q\cdot(q+N)^N$

So:

$$\Omega = \left(\frac{q+N}{q}\right)^q\left(\frac{q+N}{N}\right)^N\sqrt{\frac{N}{2\pi q(q+N)}}$$

as desired.

2.22 a. The number of energy units in A can be 0, 1, 2 ... 20, corresponding to 20, 19, 18 ... 0 in B. There are 21 such possible arrangements.

b. Using our formula from 2.18:

$$\Omega_{total} = \left(\frac{2N+2N}{2N}\right)^{2N}\left(\frac{2N+2N}{2N}\right)^{2N}\sqrt{\frac{2N}{2\pi\cdot 2N(2N+2N)}}$$

if we regard the two solids as a combined system.

$$\Omega_t = 2^{2N}\,2^{2N}\sqrt{\frac{1}{8\pi N}} = \frac{2^{4N}}{\sqrt{8\pi N}}$$

c. $$\Omega_A = \left(\frac{N+N}{N}\right)^{N}\left(\frac{N+N}{N}\right)^{N}\sqrt{\frac{N}{2\pi\cdot N(N+N)}}$$

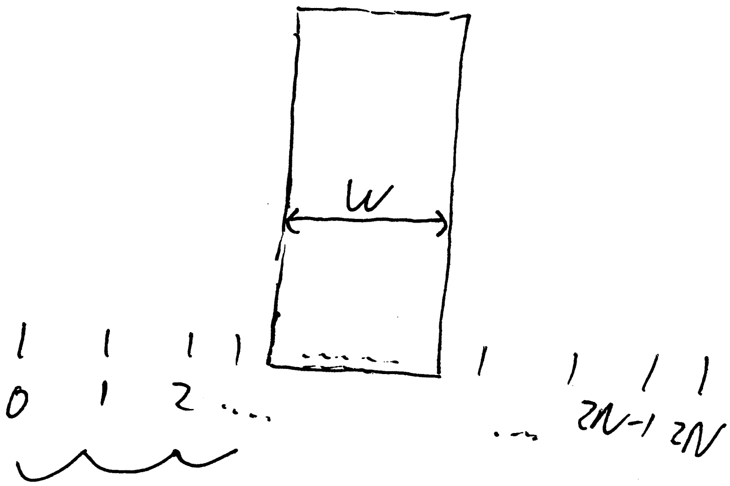
$$= 2^N 2^N\sqrt{\frac{1}{4\pi N}} = \frac{2^{2N}}{\sqrt{4\pi N}}$$

in the most likely configuration.
(Half in A, half in B)

$$\Omega_{most\,likely} = \Omega_A\cdot\Omega_B = \Omega_A^2 = \frac{2^{4N}}{4\pi N}$$

d. $\text{width} \cdot \Omega_{\text{most likely}} \approx \Omega_{\text{total}}$

$$W = \frac{\Omega_{\text{total}}}{\Omega_{\text{most likely}}} = \frac{2^{4N}/\sqrt{8\pi N}}{2^{4N}/4\pi N} = \sqrt{2\pi N}$$



2N+1 tick marks/possible macrostates

$\Rightarrow$ 2N intervals.

$$\text{Fractional width} = \frac{W}{2N} = \sqrt{\frac{\pi}{2N}}$$

2.24 a.

Ω_{max} occurs when $N_\uparrow = N_\downarrow$

$$\Omega_{max} = \frac{N!}{N_\uparrow! N_\downarrow!} = \frac{N!}{\left(\frac{N}{2}\right)!^2} \approx \frac{N^N e^{-N}\sqrt{2\pi N}}{\left(\left(\frac{N}{2}\right)^{N/2} e^{-N/2}\sqrt{2\pi \frac{N}{2}}\right)^2}$$

$$= 2^N \sqrt{\frac{2}{\pi N}}$$

b. $$\Omega = \frac{N!}{N_\uparrow! N_\downarrow!} \approx \frac{N^N e^{-N}\sqrt{2\pi N}}{N_\uparrow^{N_\uparrow} e^{-N_\uparrow}\sqrt{2\pi N_\uparrow} \cdot N_\downarrow^{N_\downarrow} e^{-N_\downarrow}\sqrt{2\pi N_\downarrow}}$$

$$= \frac{N^N}{N_\uparrow^{N_\uparrow} N_\downarrow^{N_\downarrow}} \sqrt{\frac{N}{2\pi N_\uparrow N_\downarrow}}$$

If we define x as a deviation from the state $N_\uparrow = N_\downarrow$, we have:

$$N_\uparrow = \left(\frac{N}{2} + x\right), \quad N_\downarrow = \left(\frac{N}{2} - x\right).$$

$$\Omega \approx \frac{N^N}{\left(\frac{N}{2}+x\right)^{\frac{N}{2}+x}\left(\frac{N}{2}-x\right)^{\frac{N}{2}-x}} \sqrt{\frac{N}{2\pi\left(\frac{N}{2}+x\right)\left(\frac{N}{2}-x\right)}}$$

$$= \frac{N^N}{\left[\left(\frac{N}{2}\right)^2 - x^2\right]^{N/2}\left(\frac{N}{2}+x\right)^x\left(\frac{N}{2}-x\right)^{-x}} \sqrt{\frac{N}{2\pi\left[\left(\frac{N}{2}\right)^2 - x^2\right]}}$$

$$\ln \Omega = N \ln N - \frac{N}{2} \ln\left[\left(\frac{N}{2}\right)^2 - x^2\right] - x \ln\left[\frac{N}{2} + x\right]$$
$$+ x \ln\left[\frac{N}{2} - x\right] + \ln\left(\sqrt{\frac{N}{2\pi}}\right) - \frac{1}{2} \ln\left[\left(\frac{N}{2}\right)^2 - x^2\right]$$

If $x << N$, we can use the approximation $\ln(1+x) \approx x$ for small x.

$$\ln\left[\left(\frac{N}{2}\right)^2 - x^2\right] = \ln\left[\left(\frac{N}{2}\right)^2 \cdot \left(1 - \left(\frac{2x}{N}\right)^2\right)\right]$$
$$= \ln\left(\left(\frac{N}{2}\right)^2\right) + \ln\left(1 - \underbrace{\left(\frac{2x}{N}\right)^2}_{\text{small}}\right)$$
$$\approx 2 \ln \frac{N}{2} - \left(\frac{2x}{N}\right)^2$$

Similarly,

$$\ln\left[\frac{N}{2} \pm x\right] \approx \ln\left(\frac{N}{2}\right) \pm \frac{2x}{N}$$

So we can say

$$\ln \Omega \approx N \ln N - N \ln\left(\frac{N}{2}\right) + \frac{2x^2}{N} - x \ln\left(\frac{N}{2}\right) - \frac{2x^2}{N}$$
$$+ x \ln\left(\frac{N}{2}\right) - \frac{2x^2}{N} + \ln\left(\sqrt{\frac{N}{2\pi}}\right) - \ln\left(\frac{N}{2}\right) + \frac{2x^2}{N^2}$$

$$\ln \Omega = N \ln 2 - \frac{2x^2}{N} + \ln\sqrt{\frac{2}{\pi N}} - \frac{2x^2}{N^2}$$

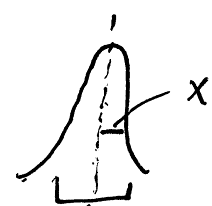
This term is much smaller than the others since it is being divided by N^2.

$$\ln \Omega \approx N \ln 2 - \frac{2x^2}{N} + \ln\sqrt{\frac{2}{\pi N}}$$

$$\Omega = 2^N \sqrt{\frac{2}{\pi N}} \exp\left[-\frac{2x^2}{N}\right]$$

c. Falls to $\frac{1}{e}$ of its max value

when $\frac{2x^2}{N} = 1$. $x = \sqrt{\frac{N}{2}}$.



Full width

Full width $= 2x = \sqrt{2N}$

d. For $N = 10^6$, half-width $= \sqrt{500{,}000} \approx 700$

1000 is not too far outside this range... it is plausible. However, 10,000 is way outside that range. It would be extremely unlikely.

2.33 $\frac{V}{N} = \frac{kT}{p} = \frac{1.38 \cdot 10^{-23} \, J/K \cdot 300\,K}{10^5 \text{ Pascals}} = 4.14 \cdot 10^{-26} \, m^3$

$\frac{U}{N} = \frac{3}{2} kT = \frac{3}{2} (1.38 \cdot 10^{-23} \, J/K) \, 300\,K = 6.21 \cdot 10^{-21} \, J$

Mass = m = mass of 1 argon atom

$= 6.64 \cdot 10^{-26} \, kg$

$\frac{V}{N} \left(\frac{4\pi m U}{3 N h^2} \right)^{3/2} = (4.14 \cdot 10^{-26}) \left(\frac{4\pi \cdot 6.64 \cdot 10^{-26} \cdot 6.21 \cdot 10^{-21}}{3(6.63 \cdot 10^{-34})^2} \right)^{3/2}$

$= 1.02 \cdot 10^7$

$S = R\left(\ln(1.02 \cdot 10^7) + \frac{5}{2}\right) = 155 \, J/K$

Helium is lighter, resulting in a smaller value of m. The area of the momentum hypersphere is smaller for helium. (See Schroeder Pg. 70)

2.37 Number of B molecules = $x N_{total}$

When the partition is removed,

$$\frac{V_{fB}}{V_{oB}} = \frac{1}{x} \Rightarrow \Delta S_B = xNk \ln\left(\frac{1}{x}\right) = -Nkx\ln x$$

by the Sackur-Tetrode eqn.

Number of A molecules = $(1-x) N_{total}$

$$\frac{V_{fA}}{V_{oA}} = \frac{1}{1-x} \Rightarrow \Delta S_A = (1-x)Nk \ln\left(\frac{1}{1-x}\right)$$

$$= -Nk(1-x)\ln(1-x)$$

$$\Delta S_{mixing} = \Delta S_A + \Delta S_B = \boxed{-Nk\left[x\ln x + (1-x)\ln(1-x)\right]}$$

as desired

If $x = \frac{1}{2}$,

$$\Delta S_{mixing} = -Nk\left[\frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{1}{2}\right)\right] = Nk\ln 2$$

Here we call N the total number of molecules. In 2.54 that was defined as $2N$. So the answers match.