Homework 6 Solutions

Schroeder 3.5, 3.8, 3.9, 3.10, 3.14, 3.25 a.-e.

Carter 7-4, 7-8, 7-11, 7-13, 7-15

Schroeder:

3.5 $\Omega = \left(\frac{eN}{q}\right)^q$ $\quad S = k \ln\left(\frac{eN}{q}\right)^q = kq \ln\left(\frac{eN}{q}\right)$

$S = kq(\ln e + \ln N - \ln q) = kq(\ln N - \ln q + 1)$

$U = q \cdot \varepsilon$, where ε is the size of each energy unit.

$S = k\frac{U}{\varepsilon}(\ln N - \ln U + \ln \varepsilon + 1)$

$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{k}{\varepsilon}\left[\ln\left(\frac{N\varepsilon}{U}\right) + 1\right] + \frac{kU}{\varepsilon} \cdot \frac{-1}{U} = \frac{k}{\varepsilon}\ln\left(\frac{N\varepsilon}{U}\right)$

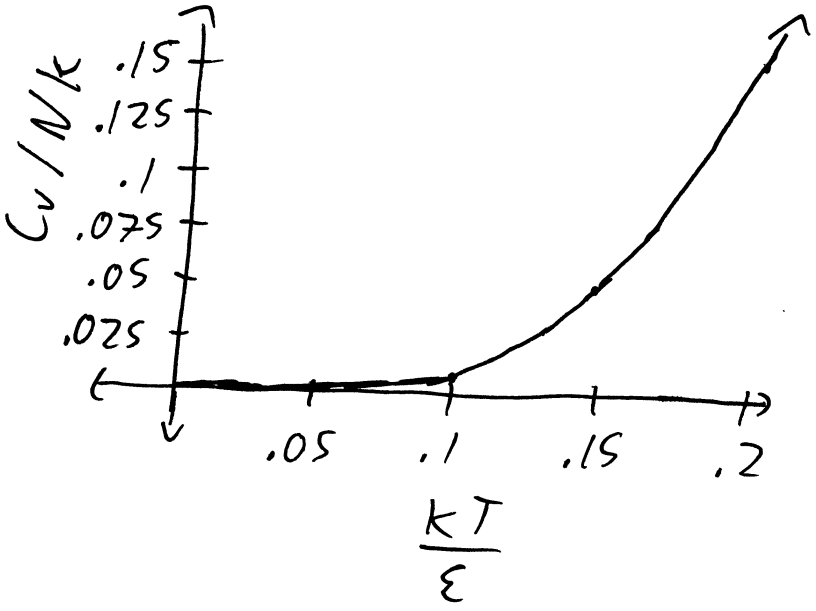
Solving for U yields

$U = N\varepsilon e^{-\varepsilon/kT}$

U @ $T = 0 \rightarrow 0$, which makes sense.

3.8 $C_v = N\varepsilon \frac{d}{dT} e^{-\varepsilon/kT}$

$= N\varepsilon \frac{\varepsilon}{kT^2} e^{-\varepsilon/kT} = Nk\left(\frac{\varepsilon}{kT}\right)^2 e^{-\varepsilon/kT}$



... approximately.

3.9 Multiplicity $= 2^N$ (like 2-state magnet)

$S = k \ln(2^N) = \boxed{Nk \ln 2}$

For $N = 6.02 \cdot 10^{23}$, $\frac{S}{k} = 4.2 \cdot 10^{23}$

$S = 5.8 \text{ J/K}$

3.10 a. $\Delta S = \frac{Q}{T} = \frac{mL}{T} = \frac{30\,g \; 333\,J/g}{273\,K}$

$= \boxed{36.6\,J/K}$

b. $\Delta S = \int_{T_i}^{T_f} \frac{C\,dT}{T} = C \ln\left(\frac{T_f}{T_i}\right)$

$= (30\,g)\left(4.186\,\frac{J}{g \cdot K}\right) \ln\left(\frac{298}{273}\right) = \boxed{11.0\,\frac{J}{K}}$

c. Heat lost by kitchen = heat gained by water!

$\Delta S = \frac{Q_{kitchen}}{T_{kitchen}} = \frac{mL + mc\Delta T}{T_{kitchen}}$

$= \frac{-(30\,g)(333\,J/g) - (30\,g) \cdot 4.186\,\frac{J}{g \cdot K} \cdot 25K}{298\,K}$

$= \boxed{-44.1\,J/K}$

d. $\Delta S_{universe} = 36.6 + 11.0 - 44.1$

$= \boxed{3.5\,J/K}$

3.14 $\Delta S = \int_0^{T_f} \frac{C_v}{T}\, dT = \int_0^{T_f} \frac{aT + bT^3}{T}\, dT$

$S(T_f) = a T_f + \frac{b}{3} T_f^3 + C$

$S(0) = 0 \Rightarrow C = 0$

$S(T) = a T + \frac{b}{3} T^3$

$S(1\,K) = \left(0.00135 \frac{J}{K^2}\right) \cdot (1\,K) + \frac{1}{3}\left(.000248 \frac{J}{K^4}\right)(1\,K)^3$

$= 0.00136\ J/K$

$S(10\,K) = 0.0218\ J/K$

$\frac{S(1\,K)}{k} = 9.8 \cdot 10^{19}$

$\frac{S(10\,K)}{k} = 1.58 \cdot 10^{21}$

3.25

a. $S = k \ln \Omega$

$= k \ln \left(\frac{q+N}{q}\right)^q + k \ln \left(\frac{q+N}{N}\right)^N$

$$\boxed{S = kq \ln \left(\frac{q+N}{q}\right) + kN \ln \left(\frac{q+N}{N}\right)}$$

(Factors of order $\sqrt{N}$ or $\sqrt{q}$ are considered small compared to factors of order q & N)

b. $\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{\partial q}{\partial U}\frac{\partial S}{\partial q} = \frac{1}{\varepsilon}\frac{\partial S}{\partial q}$ (since $q\varepsilon = U$)

$= \frac{k}{\varepsilon}\frac{\partial}{\partial q}\left(q \ln(q+N) - q \ln q + N \ln(q+N) - N \ln N\right)$

$= \frac{k}{\varepsilon}\left(\ln(q+N) + \frac{q}{q+N} - \ln q - \frac{q}{q} + \frac{N}{q+N} + 0\right)$

$= \frac{k}{\varepsilon}\left(\ln\left(\frac{q+N}{q}\right) + \frac{q+N}{q+N} - \frac{q}{q}\right)$

$$\frac{1}{T} = \frac{k}{\varepsilon} \ln\left(1 + \frac{N}{q}\right)$$

$$T = \frac{\varepsilon}{k \ln\left(1 + \frac{N}{q}\right)} = \boxed{\frac{\varepsilon}{k \ln\left(1 + \frac{N\varepsilon}{U}\right)}}$$

c. Solving the above equation for U in terms of T yields

$$U = \frac{N\varepsilon}{e^{\varepsilon/kT} - 1}$$

$$C = \frac{dU}{dT} = \frac{-N\varepsilon}{(e^{\varepsilon/kT} - 1)^2} \cdot \frac{d}{dT}\left(e^{\varepsilon/kT}\right)$$

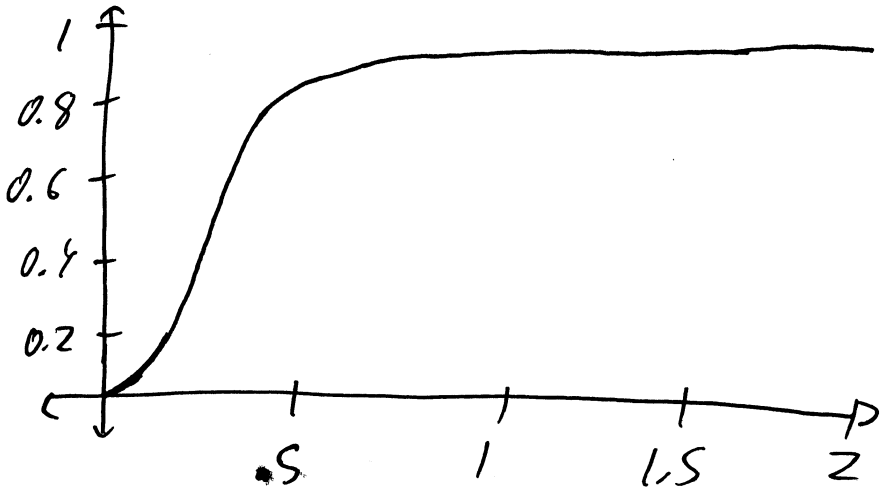
$$\boxed{C = \frac{N\varepsilon^2}{kT^2} \cdot \frac{e^{\varepsilon/kT}}{(e^{\varepsilon/kT} - 1)^2}}$$

d. When $kT >> \varepsilon$, $e^{\varepsilon/kT} \approx 1 + \frac{\varepsilon}{kT}$

$$C \simeq \frac{N\varepsilon^2}{kT^2} \frac{\left(1 + \frac{\varepsilon}{kT}\right)}{\left(\frac{\varepsilon}{kT}\right)^2} \simeq \frac{N\varepsilon^2}{kT^2} \frac{1}{\left(\frac{\varepsilon}{kT}\right)^2} = Nk$$

$\boxed{C \simeq Nk}$ Agrees with equipartition theorem, since each oscillator counts as 2 degrees of freedom (1 kinetic, 1 potential).

e. $\frac{C}{Nk} = \frac{e^{1/t}}{t^2(e^{1/t}-1)^2}$



— approximately.

Approximately similar to the curves in Figure 1.14

At high temperatures, the heat capacities are higher than theoretical values because we are measuring C_p instead of C_v.

At low T, the curve predicted by the Einstein model is flatter than the data as it approaches the origin.

C_v reaches half of its equilibrium value at $kT \approx \frac{\varepsilon}{3}$. For lead, $T_{1/2} = 22K \Rightarrow$ $\varepsilon = .0057 eV.$ For aluminum, $T_{1/2} = 100K \Rightarrow$ $\varepsilon = .026 eV.$ For diamond, $T_{1/2} = 460 \Rightarrow$ $\varepsilon = 0.12 eV$

7.4 a. $\Delta S_{body} = \int_{200}^{400} \frac{1\,kg \cdot 10\,J/kg\cdot K}{T}\,dT$

$= 10 \ln\left(\frac{400}{200}\right) = \boxed{6.93\,J/K}$

$\Delta S_{reservoir} = \frac{-1\,kg \cdot 10\,J/kg\cdot K \cdot 200\,K}{400\,K}$

$= \boxed{-5.0\,J/K}$

b. $\Delta S_{body} = \int_{200}^{100} \frac{10}{T}\,dT = 10 \ln\left(\frac{100}{200}\right) = \boxed{-6.93\,J/K}$

$\Delta S_{reservoir} = \frac{1 \cdot 10 \cdot 100K}{100K} = \boxed{10\,J/K}$

C. a. $\Delta S_{universe} = 6.93 - 5.0 = \boxed{1.93\,J/K}$

b. $\Delta S_{universe} = -6.93 + 10 = \boxed{3.07\,J/K}$

7.-8 a. $\Delta Q_{net} = 0$

$$mc_p(T_f - T_1) + mc_p(T_f - T_2) = 0$$

$$2T_f = T_1 + T_2$$

$$T_f = \frac{T_1 + T_2}{2}$$

$$\Delta S_1 = \int_{T_1}^{T_f} \frac{mc_p}{T} dT = mc_p \ln\left(\frac{T_f}{T_1}\right)$$

$$\Delta S_2 = \int_{T_2}^{T_f} \frac{mc_p}{T} dT = mc_p \ln\left(\frac{T_f}{T_2}\right)$$

$$\Delta S_{universe} = mc_p \ln\left(\frac{T_f}{T_1}\right) + mc_p \ln\left(\frac{T_f}{T_2}\right)$$

$$= mc_p \ln\left(\frac{T_f^2}{T_1 T_2}\right) = mc_p \ln\left(\frac{T_f^2}{\sqrt{T_1 T_2}^2}\right)$$

$$= 2mc_p \ln\left(\frac{T_f}{\sqrt{T_1 T_2}}\right) = \boxed{2mc_p \ln\left(\frac{T_1 + T_2}{2\sqrt{T_1 T_2}}\right)}$$

b. True if $T_1 + T_2 > 2\sqrt{T_1 T_2}$ $\Rightarrow$ $T_1 + T_2 + 2\sqrt{T_1 T_2} - 2\sqrt{T_1 T_2} > 2\sqrt{T_1 T_2}$

$$T_1 - 2\sqrt{T_1 T_2} + T_2 + 2\sqrt{T_1 T_2} > 2\sqrt{T_1 T_2}$$

$$\left(\sqrt{T_1} - \sqrt{T_2}\right)^2 + 2\sqrt{T_1 T_2} > 2\sqrt{T_1 T_2}$$

Since $\left(\sqrt{T_1} - \sqrt{T_2}\right)^2 > 0$, this is clearly a true statement.

7-11

$$T\,ds = c_v\,dT + T\left(\frac{\partial P}{\partial T}\right)_v dv$$

$$ds = \frac{c_v}{T}\,dT + \left(\frac{\partial P}{\partial T}\right)_v dv$$

$$\int ds = \int \frac{c_v}{T}\,dT + \int \left(\frac{\partial P}{\partial T}\right)_v dv$$

$$P = \frac{RT}{v-b} - \frac{a}{v^2}\,; \quad \left(\frac{\partial P}{\partial T}\right)_v = \frac{R}{v-b}$$

$$s - s_0 = c_v \ln\left(\frac{T}{T_0}\right) + R \ln\left(\frac{v_0 - b}{v_0 - b}\right)$$

$$s = c_v \ln T + R \ln(v-b) - \ln(v_0 - b) + s_0 - \ln T_0$$

Might as well rename these three constants "s_0"

$$s = c_v \ln T + R \ln(v-b) + s_0$$

7-13 a. $\chi = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T \qquad \chi_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S$

Setting $ds = 0$ in equation (7.12) yields:

$$\left(\frac{\partial V}{\partial P}\right)_S = -\frac{\beta V}{C_P} \cdot \frac{C_V \chi}{\beta} = -V\chi\frac{C_V}{C_P}$$

$$\chi_S = -\frac{1}{V} \cdot -V\chi\frac{C_V}{C_P} = \chi\frac{C_V}{C_P}$$

But, from (7.19), we have

$$C_V = C_P - \frac{TV\beta^2}{\chi} \Longrightarrow \frac{C_V}{C_P} = 1 - \frac{TV\beta^2}{C_P\chi}$$

$$\chi_S = \chi \cdot \frac{C_V}{C_P} = \chi - \frac{TV\beta^2}{C_P}$$

Therefore

$$\chi - \chi_S = \chi - \left(\chi - \frac{TV\beta^2}{C_P}\right)$$

$$= \frac{TV\beta^2}{C_P}, \text{ as desired.}$$

6. $PV = RT \qquad P = \frac{RT}{V}$

For an ideal gas

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{V}\left(\frac{R}{P}\right) = \frac{1}{T} \; ; \; C_P = \frac{5}{2}R$$

$$\chi - \chi_S = \frac{TV}{\frac{5}{2}R} \cdot \frac{1}{T^2} = \frac{2}{5} \cdot \frac{V}{RT} = \boxed{\frac{2}{5} \cdot \frac{1}{P}}$$

7-15 a. Clint Eastwood:

$$\eta = \left(\frac{\partial T}{\partial v}\right)_u = \left(\frac{\partial T}{\partial v}\right)_s + \left(\frac{\partial T}{\partial s}\right)_v \left(\frac{\partial s}{\partial v}\right)_u$$

Setting $ds = 0$ in (7.10) yields

$$\left(\frac{\partial T}{\partial v}\right)_s = -\frac{T\beta}{c_v \kappa}$$

Setting $dv = 0$ in (7.10) yields

$$\left(\frac{\partial T}{\partial s}\right)_v = \frac{T}{c_v}$$

Setting $du = 0$ in (7.9) yields

$$\left(\frac{\partial s}{\partial v}\right)_u = \frac{p}{T}$$

$$\eta = -\frac{T\beta}{c_v \kappa} + \frac{T}{c_v} \cdot \frac{p}{T} = \boxed{\frac{1}{c_v}\left(p - \frac{T\beta}{\kappa}\right)}$$

b. $$\mu = \left(\frac{\partial T}{\partial p}\right)_h = \left(\frac{\partial T}{\partial p}\right)_s + \left(\frac{\partial T}{\partial s}\right)_p \left(\frac{\partial s}{\partial p}\right)_h$$

Setting $ds = 0$ in (7.11) yields

$$\left(\frac{\partial T}{\partial p}\right)_s = [redacted] \frac{Tv\beta}{c_p}$$

Setting $dp = 0$ in (7.11) yields

$$\left(\frac{\partial T}{\partial s}\right)_p = \frac{T}{C_p}$$

Setting $dh = 0$ in the identity

$T ds = dh - v\,dp$ yields

$$\left(\frac{\partial s}{\partial p}\right)_h = -\frac{V}{T}$$

$$\mu = \frac{T v \beta}{C_p} - \frac{V}{T} \cdot \frac{T}{C_p}$$

$$\mu = \frac{V}{C_p}\left(T\beta - 1\right)$$

C. VdW Gas:

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \qquad \chi = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

$$p + \frac{a}{V^2} = \frac{RT}{V - b}$$

$$P = \frac{RT}{v-b} - \frac{a}{v^2}$$

Note that $\frac{\beta}{\kappa} = \left(\frac{\partial P}{\partial T}\right)_v$

$$\left(\frac{\partial P}{\partial T}\right)_v = \frac{R}{v-b}$$

So $\boxed{\eta = \frac{1}{C_v}\left(P - \frac{TR}{v-b}\right)}$

To find u, we still need to compute $\left(\frac{\partial v}{\partial T}\right)_P$: $\left(\frac{\partial v}{\partial T}\right)_P = -\frac{\left(\frac{\partial P}{\partial T}\right)_v}{\left(\frac{\partial P}{\partial v}\right)_T}$ (Cyclic relation)

$$= -\frac{\frac{R}{v-b}}{-\frac{RT}{(v-b)^2} + \frac{2a}{v^3}} = \frac{R(v-b)}{RT - \frac{2a(v-b)^2}{v^3}}$$

$$\left(\frac{\partial v}{\partial T}\right)_P = \frac{R(v-b)v^3}{RTv^3 - 2a(v-b)^2}$$

$$\beta = \frac{R(v-b)v^2}{RTv^3 - 2a(v-b)^2}$$

$$\mu = \frac{v}{c_p}\left(\frac{TR(v-b)v^2}{RTv^3 - 2a(v-b)^2} - 1\right)$$

For an ideal gas, $\beta = \frac{1}{T}$ & $\kappa = \frac{1}{p}$

$$\eta = \frac{1}{c_v}\left(p - T\cdot\frac{p}{T}\right) = 0 \checkmark$$

$$\mu = \frac{v}{c_p}\left(T\cdot\frac{1}{T} - 1\right) = 0 \checkmark$$